SOLUTIONS TO EXERCISES FOR

MATHEMATICS 144 — Part 3

Fall 2006

IV. Relations and functions

IV.4 : Composite and inverse functions

Exercises to work

1. Suppose that X is linearly ordered and $a, b \in X$ are distinct. Then either $a < b$ or $b < a$, and since f is strictly increasing this means that $f(a) < f(b)$ or $f(b) < f(a)$. In both cases we have $f(a) \neq f(b)$, and therefore f is 1–1.

Here is a counterexample when X is not linearly ordered. Let X be the set of all subsets of $\{0,1\}$ ordered by inclusion, let Y be the nonnegative integers, and let $f : X \to Y$ denote the number of elements in the subset $A \subset \{0,1\}$. Then f is strictly increasing, but $f[\{0\}] = f[\{1\}]$.■

2. First of all, the problem should be corrected to read, "Given a set X, let $P_0(X)$ denote the set of **nonempty** subsets of X, and define $h : P_0(A) \times P_0(B) \to P_0(A \times B)$ by $h(C, D) = C \times D$."

[Otherwise the map is not 1–1 because, say, $h(\emptyset, D) = \emptyset$.]

If $h_0(C, D) = h_0(C', D')$ then $C \times D = C' \times D'$. Suppose that $x \in C$ and $y \in D$. Then $(x, y) \in C \times D = C' \times D'$ implies that $x \in C'$ and $y \in D'$, so that $C \subset C'$ and $d \subset D'$. Conversely, if $x \in C'$ and $y \in D'$, then $(x, y) \in C' \times D' = C \times D$ implies that $x \in C'$ and $y \in D'$, so that $C \subset C'$ and $d \subset D'$. Therefore $C = C'$ and $D = D'$. To see that h_0 is not onto, let $A = B = \{0,1\}$ and note that $E = A \times B - \{(1,1)\}$ is not in the image of h_0, which consists of the sets $\{x, y)\}$, $\{x\} \times B$, $A \times \{y\}$, and $A \times B$. Note that there are 9 sets in the image and 15 sets in the codomain.■

3. (a) This map is injective.■

(b) This map is not injective.■

(c) This map is not injective.■

4. (a) This map is bijective.■

(b) This map is not bijective; it is neither injective nor surjective.■

(c) This map is not bijective from the reals to themselves, but it does define a bijection from $\mathbf{R} - \{-2\}$ to $\mathbf{R} - \{1\}$.■

(d) This map is bijective.■

5. Suppose that f is a formal monomorphism; we need to show that f is also injective. In other words, if $x \neq y$ we need to prove that $f(x) \neq f(y)$. Let $X, Y : \{1\} \to A$ be the functions such that $X(1) = x$ and $Y(1) = y$. Then $f \circ X(1) = f(x)$ and $f \circ Y(1) = f(y)$. It suffices to show that $f \circ X \neq f \circ Y$. Assume the contrary, so that $f \circ X = f \circ Y$. Since f is a formal monomorphism this would imply that $X = Y$, which would mean that $x = y$, a contradiction. Therefore a formal monomorphism is injective.

Conversely, suppose that f is injective, and let $g, h : C \to A$ satisfy $f \circ g = f \circ h$. Then for all $z \in C$ we have $f(g(z)) = f(h(z))$, and since f is injective this means that $g(z) = h(z)$ for all z. Therefore $g = h$; since the latter were arbitrary pairs of functions satisfying $f \circ g = f \circ h$, it follows that f is a formal monomorphism.■

6. Suppose that f is surjective, and suppose that $g, h : B \to D$ satisfy $g \circ f = h \circ f$; we want to show that $g = h$. Let $b \in B$, and choose $a \in A$ such that $f(a) = b$. Then $g(b) = g(f(a)) = h(f(a)) = h(b)$, and since b was an arbitrary element of B it follows that $g = h$. Therefore f is a formal epimorphism.

To prove the other implication, suppose that f is not surjective. Define $g, h : B \to \{0, 1\}$ as follows: Set $g(b) = 1$ for all b, and set $h(b) = 1$ if $b = f(a)$ for some a and set $h(b) = 0$ otherwise. Then $g \neq h$ because f is not surjective, but for all $a \in A$ we have $g(f(a)) = h(f(a)) = 1$ so that $g \circ f = h \circ f$, and hence f is not a formal epimorphism.■

7. (*b*) We always have $f[A \cap B] \subset f[A] \cap f[B]$, so we really need to prove that $f[A \cap B] \supset f[A] \cap f[B]$ for all A and B if and only if f is 1–1. ($\Longrightarrow$) Let $A = \{a\}$ and $B = \{b\}$ where $A \neq B$. Then $A \cap B = \emptyset$, so the condition on images implies

$$\emptyset \quad = \quad f[A \cap B] \quad \supset f[A] \cap f[B] \quad = \quad \{f(a)\} \cap \{f(b)\}$$

which can only happen if $f(a) \neq f(b)$. Thus f is 1–1 if $f[A \cap B] \supset f[A] \cap f[B]$ for all A and B. ($\Longleftarrow$) Suppose now that $f[A \cap B] \not\supset f[A] \cap f[B]$ for some A and B. Note first that both A and B must be nonempty because $f[A \cap B] = f[A] \cap f[B]$ if either A or B is empty. Then by noncontainment there is some $y \in f[A] \cap f[B]$ such that $y \notin f[A \cap B]$. This means there are $a \in A$ and $b \in B$ such that $y = f(a) = f(b)$, but $a \notin B$ and $b \notin A$. The preceding conditions mean that $a \neq b$, and since $f(a) = f(b)$ it follows that f is not 1–1.■

(*b*) ($\Longrightarrow$) Let $A = \{a\}$ and let $b \neq a$. Then $f(b) \in f[X - A] \subset Y - f[A] = Y - \{f(a)\}$ implies that $f(b) \neq f(a)$, and therefore f is 1–1. ($\Longleftarrow$) Suppose now that there is a subset $A \subset X$ such that $f[X - A] \not\subset Y - f[A]$. Note first that $A \neq \emptyset$ because we do have set-theoretic containment (in fact, equality!) if $A = \emptyset$. It follows that there is some y such that $y \in f[X - A]$ but $y \notin Y - f[A]$. The second condition is equivalent to $y \in f[A]$. We have thus shown that there are $u, v \in X$ such that $u \in A$ and $v \notin A$ but $y = f(u) = f(v)$. Therefore f is not 1–1.■

(*d*) ($\Longrightarrow$) Let $y \in Y$. If $y \in f[A]$ then $y = f(a)$ for some $a \in A$, while if $y \in Y - f[A] = f[X - A]$ then $y = f(b)$ for some $b \in X - A$. In both cases y lies in the image of f. ($\Longleftarrow$) Suppose that f is onto. If $y \in Y - f[A]$, then $y = f(x)$ for some x, but since $y \notin f(A)$ it follows that $x \in X - A$ and hence $y \in\subset f[X - A]$. Therefore $Y - f[A] \subset f[X - A]$.■

8. Follow the hint. Let $z \in A$ be arbitrary, and define $g : B \to A$ in cases: If $b = f(a)$ for some $a \in A$, set $g(b) = a$; since f is 1–1 there is only one possible choice for a, so this defines the function on all points of the given type. If $b \neq f(a)$ for any choice of a, then set $g(b) = z$. Then $g(f(a)) = a$ by construction, so that $g \circ f = \mathrm{id}_A$.■

9. Suppose we are given $h, k : C \to A$ such that $f \circ h = f \circ k$. Compose both sides with g:

$$g \circ f \circ h \quad = \quad \mathrm{id}_A \circ h \quad = \quad h$$

$$g \circ f \circ k \quad = \quad \mathrm{id}_A \circ k \quad = \quad k$$

Since $f \circ h = f \circ k$ the expressions on the left hand sides of both lines are equal, and hence the same is true for the expressions on the right hand sides. Hence $h = k$ and f is a monomorphism.

To show that g is an epimorphism, suppose we are given $u, v : B \to D$ such that $u \circ g = v \circ g$, and compose both sides with f:

$$u \circ g \circ f \quad = \quad u \circ \mathrm{id}_A \quad = \quad u$$

$$v \circ g \circ f \quad = \quad v \circ \mathrm{id}_A \quad = \quad v$$

As before the left hand sides of both lines are equal, so the right hand sides are too, and hence $u = v$ so that g is an epimorphism.■

10. This is really the same as the previous exercise if we interchange the roles of A and B and of f and g. The whole point is that we have two maps whose composite QP is the identity, and under these conditions P is a monomorphism and Q is an epimorphism.■

11. The simplest example is the linear mapping $L(t)$ sending $t \in [0, 1]$ to $a + t(b - a)$. The mapping is 1–1 because $L(s) = L(t)$ implies $a + s(b - a) = a + t(b - a)$ and since $a \neq b$ one can solve this equation to conclude that $s = t$. To see that the mapping is onto, one need only check that if $a \leq c \leq b$ and

$$t \quad = \quad \frac{c - a}{b - a}$$

then $0 \leq t \leq 1$ and $L(t) = c$. The 1–1 correspondence L is definitely not unique. Let K be any 1–1 correspondence of $[0, 1]$ with itself that is not the identity; for example, one could take $K(t) = t^2$, whose inverse is the square root function. Then $L \circ K$ is a different 1–1 correspondence between $[0, 1]$ and $[a, b]$.■

12. Let $\mathbf{N}$ be the set of nonnegative integers, let $f : \mathbf{N} \to \mathbf{N}$ send x to $x + 1$, and let $g : \mathbf{N} \to \mathbf{N}$ send x to $|x - 1|$. Then f is not surjective because $0 \neq f(x)$ for any x, and g is not injective because $g(2) = g(0)$, but $g \circ f$ is the identity on $\mathbf{N}$. This yields the example for the first part of the problem.

As noted there are four subcases to the second part of the problem:

Is f injective if $g \circ f$ is bijective? YES. Suppose that x and y lie in the domain of f and $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$, and since $g \circ f$ is bijective it follows that $x = y$.

Is f surjective if $g \circ f$ is bijective? NOT NECESSARILY. Consider the example constructed for the first part of the problem.

Is g surjective if $g \circ f$ is bijective? YES. If z is in the codomain of g, then the bijectivity of $g \circ f$ implies that $z = g(f(w))$ for some w, and thus we know that g maps $f(w)$ to z.

Is g injective if $g \circ f$ is bijective? NOT NECESSARILY. Consider the example constructed for the first part of the problem.■

13. In both cases the idea is to write $y = f(x)$ and solve for x in terms of y.

If $f(x) = 3x - 1$, then $y = 3x - 1$ implies that $x = \frac{1}{3}(x + 1)$, so the right hand side gives the inverse function.

If $f(x) = x/(1 + |x|)$, then solving for x in terms of y splits into two cases depending upon whether $x \geq 0$ or $x \leq 0$. Note that these are equivalent to $y \geq 0$ and $y \leq 0$ by the definition of f. If $x \geq 0$ then we have

$$y \quad = \quad \frac{x}{1 + x} \quad \Longrightarrow \quad x \quad = \quad \frac{y}{1 - y}$$

while if $x \leq 0$ we have

$$y \quad = \quad \frac{x}{1 + x} \quad \Longrightarrow \quad x \quad = \quad \frac{y}{1 + y}$$

so in either case we have

$$x \quad = \quad \frac{y}{1-|y|}$$

as the formula for the inverse function.■

14. Let $n = \mathbf{int}(x)$. There are two cases depending upon whether $n \leq x < n + \frac{1}{2}$ or $n + \frac{1}{2} \leq x < n + 1$.

In the first case we have $\mathbf{int}(x) = \mathbf{int}\left(x + \frac{1}{2}\right) = n$ and since $2n \leq 2x < 2n + 1$ we also have $\mathbf{int}(2x) = 2n$. Therefore $\mathbf{int}(2x)$ and $\mathbf{int}(x) + \mathbf{int}\left(x + \frac{1}{2}\right)$ are both equal to $2n$ in this case.

In the second case we have $\mathbf{int}(x) = n$ but $\mathbf{int}(x + \frac{1}{2}) = n + 1$ and $2n + 1 \leq 2x < 2n + 2$. Therefore $\mathbf{int}(2x)$ and $\mathbf{int}(x) + \mathbf{int}\left(x + \frac{1}{2}\right)$ are both equal to $2n + 1$ in this case.■

15. Both statements are false, and this can be seen by taking $x = y = \frac{3}{5}$. For these choices we have

$$1 \quad = \quad \mathbf{int}\left(\tfrac{3}{5} + \tfrac{3}{5}\right) \quad \neq \quad \mathbf{int}\left(\tfrac{3}{5}\right) \ + \ \mathbf{int}\left(\tfrac{3}{5}\right) \quad = \quad 0 \ + \ 0 \quad = \quad 0$$

and

$$2 \quad = \quad \mathbf{int}\left(\tfrac{6}{5}\right) \ + \ \mathbf{int}\left(\tfrac{6}{5}\right) \quad \neq \mathbf{int}\left(\tfrac{3}{5}\right) \ + \ \mathbf{int}\left(\tfrac{3}{5}\right) \ + \ \mathbf{int}\left(\tfrac{6}{5}\right) \quad = \quad 0 \ + \ 0 \ + \ 1 \quad = \quad 1\ .$$

Therefore the have values of x and y for which $\mathbf{int}(x) + \mathbf{int}(y) \neq \mathbf{int}(x + y)$ and $\mathbf{int}(x) + \mathbf{int}(y) + \mathbf{int}(x + y) \neq \mathbf{int}(2x) + \mathbf{int}(2y)$.■

16. Let $n = \mathbf{int}(x)$. There are three cases depending upon whether $n \leq x < n + \frac{1}{3}$ or $n + \frac{1}{3} \leq x < n + \frac{2}{3}$ or $n + \frac{2}{3} \leq x < n + 1$.

In the first case we have $\mathbf{int}(x) = \mathbf{int}\left(x + \frac{1}{3}\right)) = \mathbf{int}\left(x + \frac{2}{3}\right) = n$ and since $3n \leq 3x < 3n + 1$ we also have $\mathbf{int}(3x) = 3n$. Therefore $\mathbf{int}(3x)$ and $\mathbf{int}(x) + \mathbf{int}\left(x + \frac{1}{3}\right) + \mathbf{int}\left(x + \frac{2}{3}\right)$ are both equal to $3n$ in this case.

In the second case we have $\mathbf{int}(x) = n$ and $\mathbf{int}\left(x + \frac{1}{3}\right) = n$ but $\mathbf{int}\left(x + \frac{2}{3}\right) = n + 1$ and $3n + 1 \leq 3x < 3n + 2$. Therefore $\mathbf{int}(3x)$ and $\mathbf{int}(x) + \mathbf{int}\left(x + \frac{1}{3}\right) + \mathbf{int}\left(x + \frac{2}{3}\right)$ are both equal to $3n + 1$ in this case.

In the third case we have $\mathbf{int}(x) = n$ but $\mathbf{int}\left(x + \frac{1}{3}\right) = \mathbf{int}\left(x + \frac{2}{3}\right) = n + 1$ and $3n + 2 \leq 3x < 3n + 3$. Therefore $\mathbf{int}(3x)$ and $\mathbf{int}(x) + \mathbf{int}\left(x + \frac{1}{3}\right) + \mathbf{int}\left(x + \frac{2}{3}\right)$ are both equal to $3n + 2$ in this case.■

17. The important point is that if a and b are positive real numbers, then

$$a \quad < \quad b \quad \Longleftrightarrow \quad \frac{1}{a} \quad > \quad \frac{1}{b}\ .$$

Suppose now that f is strictly increasing. Then $x < y$ implies $f(x) < f(y)$, and by the preceding line and $g = 1/f$ we conclude that $g(x) > g(y)$, so that g is strictly decreasing. Conversely, if g is strictly decreasing and $x < y$, then we have $g(x) > g(y)$. Now $g = 1/f$ is true if and only if $f = 1/g$, and therefore we can use the displayed statement to conclude that $f(x) < f(y)$ and hence that f is strictly increasing.■

18. Follow the hints. Given $y \in C_0$ we would like to define H by choosing $x \in A$ so that $q_0(x) = y$ [such an x exists because q_0 is onto] and setting $H(y) = q_1(x)$.

In order to make such a definition it is necessary to show that the construction does not depend upon the choice of x; in other words, if $q_0(z) = q_0(x) = y$, then $q_1(x) = q_1(z)$. All we have at our disposal are the injectivity and surjectivity assumptions along with the identities $f = j_0 \circ q_0 = j_1 \circ q_1$. Since j_1 is injective, if we can show that $j_1 q_1(z) = j_0 q_0(x)$, then we will have $q_1(z) = q_1(x)$ as desired. But

$$j_1 q_1(z) \quad = \quad f(z) \quad = \quad j_0 q_0(z) \quad = \quad j_0(y) \quad = \quad j_0 q_0(x) \quad = \quad f(x) \quad = \quad j_1 q_1(x)$$

so we do have the necessary identity $q_1(z) = q_1(x)$. Therefore we have defined a mapping $H : C_0 \to C_1$ such that $H \circ q_0 = q_1$.

We now need to show that H is bijective. Suppose that $H(y) = H(y')$ and choose x, x' so that $y = q_0(x)$ and $y' = q_0(x')$. We then have

$$j_0(y) \quad = \quad j_0 q_0(x) \quad = \quad f(x) \quad = \quad j_1 q_1(x) \quad = \quad j_1 H(y)$$

$$j_0(y') \quad = \quad j_0 q_0(x') \quad = \quad f(x') \quad = \quad j_1 q_1(x') \quad = \quad j_1 H(y')$$

and since $H(y) = H(y')$ it follows that the expressions on both lines are equal, so that $j_0(y) = j_0(y')$. Since j_0 is injective, this implies $y = y'$ and hence H is injective. To show that H is surjective, express a typical element $z \in C_1$ as $q_1(x)$ for some x; then if $y = q_0(x)$ we have $z = H(y)$. This completes the proof that H is bijective.

All that remains is to show that H is unique. Suppose that $K : C_0 \to C_1$ also satisfies $K \circ q_0 = q_1$. Then if $y \in C_0$ and $y = q_0(x)$ we have

$$K(y) \quad = \quad K \circ q_0(x) \quad = \quad q_1(x) \quad = \quad H \circ q_0(x) \quad = \quad H(y)$$

and hence $K = H$, proving uniqueness.∎

IV.5 : Constructions involving functions

Exercises to work

1. Suppose that we are given an arbitrary function $g : A \to E \times F$ such that

$$g(a) \quad = \quad \big(\, u(a),\, v(a) \,\big)$$

for some functions $u : A \to E$ and $v : A \to F$. For each $(e, f) \in E \times F$ we then have $g(a) = (e, f)$ if and only if $u(a) = e$ and $v(a) = f$.

Let us apply this to the situation in the problem: Since $p_j(x) = d$ is equivalent to saying that x lies in the equivalence class d, it follows that $q(x) = (e, f)$ if and only if $x \in e$ and $x \in f$, which is equivalent to saying that $x \in e \cap f$.■

2. Once again, follow the hints for each part.

The correspondence $(B \times C)^A \longleftrightarrow B^A \times C^A$. As in the hint let p and q be the coordinate projections from $B \times C$ to B and C respectively. Given $f : A \to B \times C$, one has the associated pair $(p \circ f, q \circ f) \in B^A \times C^A$. This mapping is onto because one can use functions $u : A \to B$ and $v : A \to C$ to define a function $f(a) = (\, u(a),\, v(a)\,)$, and it is 1–1 because $pf' = pf$ and $qf' = qf$ imply that the first and second coordinates of $f(a)$ and $f'(a)$ are equal for all am so that f and f' are the same function.■

The correspondence $\left(C^B\right)^A \longleftrightarrow C^{B \times A}$. The hint outlines definitions of mappings

$$\Phi : C^{B \times A} \to \left(C^B\right)^A \qquad\qquad \Psi : \left(C^B\right)^A \to C^{B \times A}$$

that will be repeated in the argument. Given $f : B \times A \to C$, let Γ be its graph viewed as a subset of $B \times A \times C$, and for each $a \in A$ let Γ_a be given by taking the intersection

$$\Gamma \cap B \times \{a\} \times C$$

and projecting it down to $B \times C$ under the standard projection map $B \times \{a\} \times C \to B \times C$ which forgets the middle coordinate. We claim there is a unique function $g_a : B \to C$ whose graph is equal to $\Gamma(a)$; this amounts to checking that $\Gamma(a)$ is actually the graph of a function from B to C. Suppose we are given $b \in B$. Then there is a unique $c \in C$, namely $f(b, a)$, such that $(b, a, c) \in \Gamma$, and for this choice of c we also have $(b, c) \in \Gamma(a)$. Suppose now that $(b, c') \in \Gamma(a)$. Then by definition we have $(b, a, c') \in \Gamma$, which means that $c = c'$ and hence yields the required uniqueness statement. Therefore we have constructed a mapping Φ of the type described above.

To construct a map in the opposite direction, if we are given $g \in \left(C^B\right)^A$, then for each $a \in A$ we have a function $g(a) : B \to C$. Let $\Gamma(a)$ be the graph of $g(a)$, and Γ be the set of all ordered triples (b, a, c) such that (b, c) lies in $\Gamma(a)$. We claim that Γ is the graph of a function from $B \times A$ to C. Given $(b, a) \in B \times A$ we need to show there is a unique c such that $(b, a, c) \in \Gamma$. Existence follows because we can take $c = [g(a)](b)$. To see uniqueness, note that $(b, a, c') \in \Gamma$ implies $(b, c') \in \Gamma_a$, so that $c' = [g(a)](b)$. Thus we have the map Ψ as required.

Finally, to show there are 1–1 correspondences it is enough to verify that $\Psi \circ \Phi(f) = f$ for all f and $\Phi \circ \Psi(g) = g$ for all g. These follow because our constructions have the property that Γ is the set of all (b, a, c) such that $(b, c) \in \Gamma(a)$.■

IV.6 : Order types

Exercises to work

1. Define $f : [0,1) \cup (2,3) \to [0,2)$ by $f(x) = x$ if $x < 1$ and $f(x) = x - 1$ if $x \geq 2$. We claim f is strictly increasing (hence is 1–1), and we shall do this by considering several . separate cases. Suppose we have $u < v$. (1) If $v < 1$ then $f(u) = u < v = f(v)$. (2) If $u < 1 < 2 \leq v$ then $f(u) = u < 1 \leq v - 1 = f(v)$. (3) If $2 \leq u$ then $f(u) = u - 1 < v - 1 = f(v)$.

To complete the proof it is enough to show f is onto. This is straightforward: If $y < 1$ then $y = f(y)$, while if $y \geq 2$ then $f(y+1) = y$.■

2. The interval $[0,2]$ has the self-density property: If $u < v$ then there is some w such that $u < w < v$. On the other hand, $[0,1] \cup [2,3]$ does not because there is no w in this set such that $1 < w < 2$ (it lies in the reals, where such a w exists, but we are only interested in elements of the given partially ordered set here and not in any larger partially ordered set that might contain it). Since one partially ordered set has the self-density property but the other does not, they cannot have the same order type.■

3. Write $X = P(A)$, where A is an infinite set, and let $a \in A$. Then X does not have the self-density property because there is no subset $B \subset A$ strictly between $\emptyset$ and $\{a\}$.

Turning to Y, suppose we have polynomials f and g such that $f < g$. Before going further, we should stress what this means: We have $f(x) \leq g(x)$ for all real x and there is some c such that $f(c) < g(c)$; in particular, it does **NOT** mean that $f(x) < g(x)$ for all x.

In any case let $D = g - f$ so that $h > 0$, and consider $h = f + \frac{1}{2}D$. Then direct computation shows have $f \leq h \leq g$ (*i.e.*, $f(x) \leq h(x) \leq g(x)$ for all x) and $f(c) < h(c) < g(c)$, so that $f < h < g$.■

4. A positive integer d divides 28 if and only if it has the form $d = 2^a 7^b$ where a and b are integers satisfying $0 \leq a \leq 2$ and $0 \leq b \leq 1$, and likewise a positive integer e divides 45 if and only if it has the form $3 = 3^a 5^b$ where a and b are integers satsifying $0 \leq a \leq 2$ and $0 \leq b \leq 1$. If we define a mapping from $D(28)$ to $D(45)$ taking $d = 2^a 7^b$ to $e = 3^a 5^b$, then this will be the required order-isomorphism (we shall not check the details explicitly here),

On the other hand, in the notes we noted that $D(15)$ is a partially ordered set with 4 elements that is not linearly ordered, and the set $D(8)$ is the linearly ordered set consisting of 1, 2, 4 and 8 (where the divisibility ordering agrees with the usual ordering!). In particular, $D(8)$ also has four elements. However, since $D(8)$ is linearly ordered but $D(15)$ is not, it follows that these two partially ordered sets cannot have the same order type.■

5. As noted in the hint, for each $x \in \mathbf{N}$ the set of all y such that $y < x$ is finite, for it is just $\{1, 2, ..., x-1\}$. On the other hand, the set of all elements of $\mathbf{N} \times \mathbf{N}$ (with the lexicographic ordering) that precede $(1,0)$ is the infinite set of all ordered pairs of the form $(0,k)$ where k can be any nonnegative integer. Thus one of the linearly ordered sets under consideration has the property

for each element a the set elements x for which $x < a$ is finite

while the other does not, and consequently they cannot have the same order type.■