Addendum to IV.3 — Comment on polar coordinates

We shall now use our approach to set theory and functions in order to analyze a problem that arises in analytic geometry and calculus.

Many intermediate or advanced treatments of polar coordinates contain a section on finding the intersection points of two plane curves given in polar coordinates. If the curves are defined by equations of the form $F(r, \theta) = 0$ and $G(r, \theta) = 0$, then some points of this type are given by the values of (r, θ) which solve both of these equations, but frequently one encounters examples where this does not yield all the common points. One example of this sort is given by the circle with equation $r = 1$ and the line with equation $\theta = 1$. The common solutions of the two equations yield the point in the plane with polar coordinates $(1, 1)_{\mathrm{POLAR}}$, but if one graphs the two curves it is also apparent that $(-1, 1)_{\mathrm{POLAR}} = (1, 1 + \pi)_{\mathrm{POLAR}}$ is also on both curves.

Sometimes calculus texts address this difficulty by suggesting that one graph the two curves to see if there are any common points that are not given by simultaneous solutions of the equations. This is usually effective, but it is neither systematic nor logically complete.

We shall use the material developed thus far in this course to give a more reliable basis for finding common points. Additional details appear in the following online document:

`http://math.ucr.edu/~res/math9C/polar-ambiguity.pdf`

In fact, we shall look at an abstract version of the polar coordinate problem. Suppose that we are given a surjective function $f : A \to B$ from one set A to a second set B; in the special case of immediate interest, the sets A and B are both equal to the real numbers and f is the standard polar coordinate map sending (r, θ) to $(r \cos \theta, r \sin \theta)$. What is the abstract version of two curves C_1 and C_2 defined by equations in polar coordinates? The equations have the form $g_1(a) = x_1$ and $g_2(a) = x_2$ for functions $g_1, g_2 : A \to X$ (where X is some set), and the abstract versions of C_1 and C_2 are the sets of all points $b \in B$ such that there are some $a \in A$ satisfying $f(a) = b$ and $g_1(a) = x_1$ (for C_1) or $f(a) = b$ and $g_2(a) = x_2$ (for C_2). This intersection includes all points $b \in B$ such that there is some $a \in A$ satisfying $f(a) = b$, $g_1(a) = x_1$ and $g_2(a) = x_2$. However, the following result, which is elementary to verify, describes ALL the possibilities:

Proposition. *In the setting of the preceding paragraph, the intersection $C_1 \cap C_2$ consists of all $b \in B$ for which there exist $a_1, a_2 \in A$ such that $f(a_1) = f(a_2) = b$, $g_1(a) = x_1$, and $g_2(a) = x_2$.*■

APPLICATION TO THE PREVIOUS EXAMPLE. In this case the equations $g_1(a) = x_1$ and $g_2(a) = x_2$ are $r - 1 = 0$ and $q - \theta = 0$. We then have $f(-1, 1) = f(1, 1 + \pi)$, and we also have $g_1(-1, 1) = 0 = g_2(1, 1 + \pi)$. Therefore the criterion in the proposition implies that $f(-1, 1) = f(1, 1 + \pi) \in C_1 \cap C_2$. Graphically it is clear that this point and $f(-1, 1)$ are the only points at which the line and circle meet, but we need to check this analytically in order to be logically complete.

Given (r, θ), the definition of polar coordinates shows that $f(r, \theta) = f(s, \varphi)$ if and only if either (i) $r = s = 0$ and the second coordinate is arbitrary, (ii) $\mathrm{s} = (-1)^m r$ and $\varphi = \theta + m\pi$. Thus we need to find all pairs (r, θ) and (s, φ) which satisfy these conditions and also satisfy $r = 1$ and $\varphi = 1$. If $r = 1$, then $s = \pm 1$, so the first option does not apply. But this means that $1 = r = (-1)^m s$ and

$1 = \varphi = \theta + m\pi$, so that $\theta = 1 - m\pi$ and $s = (-1)^m$. Therefore every intersection point has the form $f(1, 1 - m\pi)$. Since the latter is equal to $f(1, 1)$ if m is even and to $f(1, 1 + \pi)$ if m is odd, it follows that the only possible intersection points of the line and circle are $f(1, 1)$ and $f(1, 1 + \pi)$; we already know that both points lie in the intersection, so therefore we know that the line and circle meet precisely in the two points $f(1, 1)$ and $f(1, 1 + \pi)$.