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Mathematics 144, Fall 2006, Examination 2

Answer Key

1. [*30 points*] (a) Is the set of all points $(x, y) \in \mathbf{R} \times \mathbf{R}$ such that $y^2 = e^x$ the graph of a function? Give reasons for your answer.

(b) Let $f : \mathbf{R} \to \mathbf{R}$ be the function $f(x) = 4x + 6$. Let A and B be the closed intervals $[1, 2]$ and $[-2, 10]$ respectively. Find the image $f[A]$ and the inverse image $f^{-1}[B]$.

(c) Let $f : A \to B$ and $g : B \to C$ be onto (= surjective) functions. Prove that the composite $g \circ f$ is also onto.

SOLUTION

(a) **NO.** If $y^2 = e^x$ then also $(-y)^2 = e^x$, so in particular the set includes the points $(0, 1)$ and $(0, -1)$.

(b) The first of these is the set of all points describable as $4x + 6$ for $1 \leq x \leq 2$, which is merely the interval $[10, 14]$, and the second is the set of all x such that $-2 \leq 4x + 6 \leq 10$, which is the interval $[-2, 1]$.

(c) Let $c \in C$. Then there is some $b \in B$ such that $g(b) = c$ because g is onto. Similarly, since f is onto there is some $a \in A$ such that $b = f(a)$. It then follows that $g \circ f(a) = g(\,f(a)\,) = g(b) = c$, showing that $g \circ f$ is onto.

2. [*25 points*] (a) If $f : [0, \infty) \to [0, 1)$ is the function

$$f(x) \quad = \quad \frac{x^2}{x^2+1}$$

then f is strictly increasing and defines a 1–1 correspondence from its domain to its codomain, and hence it has an inverse function g. Find a formula for this inverse function.

(b) Let $f : \mathbf{R} \to \mathbf{R}$ be the function $f(x) = 2x - 1$ and let $g(x) = x + 1$. Evaluate $g \circ f(2) - f \circ g(2)$.

SOLUTION

(a) We need to solve the equation

$$y \quad = \quad \frac{x^2}{x^2+1}$$

for x in terms of y. The right hand side can be rewritten as

$$1 \;-\; \frac{1}{x^2+1}$$

and therefore we obtain the equation

$$1 \;-\; y \quad = \quad \frac{1}{x^2+1}$$

so that

$$x^2 \;+\; 1 \quad = \quad \frac{1}{1-y}$$

and hence

$$x^2 \quad = \quad \frac{y}{1-y}\;.$$

Thus the inverse function is given by

$$g(y) \quad = \quad \sqrt{\frac{y}{1-y}}\;.$$

(b) We have $g \circ f(x) = 2x$ and $f \circ g(x) = 2x + 1$. Therefore $g \circ f(x) - f \circ g(x) = 1$ for all x and in particular for $x = 2$.

3. [*20 points*] Prove by induction that

$$1 + 3 + \cdots + (2n-1) = n^2$$

for all $n \geq 1$.

SOLUTION

Let $\mathbf{P}_n$ be the staatement displayed above. Then $\mathbf{P}_1$ is true because both sides are equal to 1. — Now assume that $\mathbf{P}_k$ is true; we need to show that $\mathbf{P}_{k+1}$ is also true. But now

$$1 + 3 + \cdots + (2k+1) = \Big(1 + 3 + \cdots + (2k-1)\Big) + (2k+1)$$

and by the induction hypothesis (that $\mathbf{P}_k$ is true) it follows that the first term on the right hand side is equal to k^2. Therefore the right hand side is equal to $k^2 + 2k + 1 = (k+1)^2$, and consequently $\mathbf{P}_{k+1}$ is also true if $\mathbf{P}_k$ is true. Therefore, by the weak principle of finie induction we know that $\mathbf{P}_n$ is true for every $n \geq 1$.

4. [*25 points*] (a) What is the least upper bound of the closed interval $[0, 1] = \{x \in \mathbf{R} \mid 0 \leq x \leq 1\}$ in the real numbers? Give reasons for your answer.

(b) A well-ordered set is a partially ordered set which satisfies an additional condition. State this condition.

SOLUTION

(a) The answer is **1**. Every element in the set is less than or equal to 1, so 1 is an upper bound. On the other hand, since 1 belongs to the set we also know that any upper bound u must be greater than or equal to 1.

(b) Every nonempty subset has a least element.