Exercises for Unit I (General considerations**)**

I.1 : Overview of the course

(Halmos**,** *Preface***;** Lipschutz**,** *Preface***)**

<u>Questions to answer:</u>

1. Why are there safety codes mandating that the foundation of a building must meet certain structural requirements? How might one apply the same principle to the mathematical sciences?

2. Why do manufacturers invest substantial resources into studying their mass production methods? How might one apply the same principle to the mathematical sciences?

3. Emphasis on quality standards can sometimes be taken too far. For example, in a chemical laboratory one could focus on cleaning laboratory equipment to a point that it would interfere with performing the experiments that are supposed to be carried out. How might one apply the same principle to the mathematical sciences?

4. The notes mentioned that one reason for the continued study of axiomatic set theory is to test the limits to which the foundations of mathematics can be pushed. Give one or more examples outside of the mathematical sciences where manufacturers might be expected to test the limits of their products.

I.2 : Historical background and motivation

<u>Questions to answer:</u>

1. Zeno's paradoxes are based on mixing "atomic" and "continuous" models for physical phenomena too casually. The following example shows that the casual use of mixed models still happens today: Consider the problem of finding the center of mass for an object like a solid hemisphere by means of calculus. Using calculus to find the center of mass tacitly assumes that matter is continuous. On the other hand, the atomic theory of matter implies that matter is not infinitely divisible. In view of this discrepancy, what meaning should be attached to the integral formula for the center of mass for the solid hemisphere?

2. Find the gap in the following proof that the angle sum **S** of a triangle is always 180 degrees: Let **A, B, C** be the vertices of the triangle, and let **D** be a point between **B** and **C** so that Δ**ABC** is split into Δ**ABD** and Δ**ADC.** The sums of the angles in both triangles are easily computed to be equal to the sums of the angles in Δ**ABC** plus 180 degrees. But since the angle sum of a triangle is **S,** this sum is also equal to 2 **S** and hence we have 2 **S = S +** 180**,** which implies that **S** must be equal to 180. Where is the mistake in this argument? **—** This issue is relevant to the discovery of non – Euclidean geometry because Euclid's Fifth Postulate is equivalent to assuming that the angle sum of a triangle is always 180 degrees. **[*Comment*:** It will probably be helpful to draw a picture corresponding to the assertions made in the fallacious proof.**]**

3. The following example in coordinate geometry illustrates the need for adding assumptions to those in Euclid's ***Elements.*** Consider the triangle in the coordinate plane with vertices **(**0**,** 1**), (**0**,** 0**)** and **(**1**,** 0**),** and let **L** be a line passing through the midpoint of the hypotenuse, which is **(½, ½).** Show that **L** either goes through the vertex **(**0**,** 0**)** or else contains a point on one of the other two sides. It might be helpful to break this problem up into cases depending upon the slope **m** of the line, which might be equal to 1, greater than 1, undefined, less than **–** 1, or between **–** 1 and **+** 1**. —** This example reflects a general fact about triangles and lines meeting one of the sides between the two endpoints, and it is known as ***Pasch's Postulate*** after M. Pasch (1843 – 1930), who noted its significance. This statement is used in the ***Elements,*** but it is not proven and the need to assume it is not acknowledged. **[Comment:** This is not meant to denigrate the ***Elements*;** the purpose is to illustrate that the sorts of problems in Euclid's important and monumental work that were discovered in the late 19^{th} century.**]**

4. Two ordinary decks of 52 cards are shuffled, and each of 52 players is dealt one card from each deck. Explain why at least half of the players will receive two number cards (Ace through 10). **[Hint:** It might be good to break things down into four cases, depending on what sort of card an individual receives from each deck. The number of persons receiving either a number or picture card from a fixed is of course 52, and the number of persons receiving a number card from the first or second deck is 40. Of course, the number of persons in each of the four cases is also nonnegative.**]**

5. The following example shows that some care must be taken when rearranging the terms of an infinite series because different arrangements of the terms sometimes lead to different answers. Consider the standard infinite series for the natural logarithm of 2

$$\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

and suppose we rearrange the terms as follows**:**

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \ldots$$

Explain why the sum of the rearranged series is **½** $\ln 2$**. —** The general rearrangement question is discussed in pages 75 – 78 of Rudin, ***Principles of Mathematical Analysis*** (3^{rd} Ed., McGraw – Hill, New York, 1976, ISBN: 0–07–054235–X); the subsequent proof of Theorem 3.50 is also relevant to this topic, and additional examples involving the effect of rearrangement on infinite series appear on pages 73 – 74 of Rudin. There is also a discussion with examples on pages 656 – 657 and 660 of the 10^{th} Edition of

Thomas' ***Calculus***, for which bibliographic information is given in the notes. Two points especially worth noting are that the sum of a series of ***nonnegative*** terms does not change if one rearranges the terms in any manner whatsoever and the sum of any series does not change if we only rearrange finitely many terms. More generally, the sum does not change if the series is absolutely convergent (*i.e.*, the series whose terms are the absolute values of the given ones converges). Of course, the latter fails for the series considered above.

6. One important fact about power series is that they can be differentiated term by term, and the result will be the derivative of the function represented by the series. The following example shows that term–by–term differentiation of trigonometric series is not possible. Consider the expansion for the square wave function described in the notes, and consider the termwise second derivatives of both sides away from the points of discontinuity. If it were possible to perform term–by–term differentiation on this series, then one would expect that that the second derivative of the left hand side, which is zero, should be equal to the following infinite series:

$$-4/\pi\,(\sin x + 3\sin 3x + 5\sin 5x + \dots)$$

What happens to this series if, say, $x = \pi/2$?

I.3 : Selected problems

<u>*Questions to answer:*</u>

1. In the setting of the pigeonhole principle, is it possible to place conditions on **m** and **n** which guarantee that at least two locations will contain more than one element? Explain the reasons why or why not. Does the answer change if we put an upper bound on the number of objects that can be placed in a given location? Think about the case $m = 2n$ with a stipulation that no location should contain 3 or more elements.

2. Verify that the base 8 expansion of $1/3$ is $0.2525252525\dots$ by geometric series computations. Recall that in base 8 the expression $0.x_1 x_2 x_3 \dots$ (where each x_k is an integer between 0 and 7) is the sum of the quantities $x_k 8^{-k}$.

3. If x is the unique positive cube root of 2, verify that $y = 1 + x$ is an algebraic number by expressing $y^3 = p\,y^2 + q\,y + r$ for suitable integers **p, q** and **r.**

Exercises for Unit II (Basic concepts**)**

II.0 : Topics from logic

(Lipschutz, §§ 10.1 – 10.12**)**

General remarks. This section of the notes has been included mainly as background, and consequently *the exercises below are optional*. However, it probably would be useful to look through some of these items at least briefly. Logical skills play a particularly important role in this course, so a review and assessment of them is extremely worthwhile. The exercises also provide opportunities for ensuring that these skills will be sufficient for the course. Most of the exercises on the list were selected because of their significance for the sorts of proofs one encounters in a course on set theory rather than for the sake of studying symbolic logic in its own right (which is worthwhile, but outside the objectives of this course). The first group of exercises comes from the Discrete Mathematics text by Rosen, which also contains a very large number of other excellent and relevant exercises.

Problems for study.

Lipschutz : 10.8 – 10.10, 10.17 – 10.19, 10.21, 10.23 – 10.28, 10.30

Exercises to work.

Exercises from Rosen:

> ***Note.*** The first two exercises show that one can define the three standard propositional operators *and*, *or* and *not* in terms of a single operator (***Sheffer's stroke***, named after H. M. Sheffer (1883 – 1964), which is written **p|q** and means that at least one of **p** and **q** is false. Biographical information about Sheffer appears on page 26 of Rosen.

1. (Rosen, Exercise 41, p. 27) Show that **p|q** is equivalent to $\neg (p \wedge q)$.

2. (Rosen, Exercise 46, p. 27) Show that all the logical operations $\wedge$, $\vee$, $\neg$ can all be written in terms of Sheffer's stroke. [***Hint:*** Note that $\neg p$ is equivalent to **p|p**, while $p \wedge q$ is equivalent to $\neg (p|q)$, and $p \vee q$ is equivalent to $\neg ([\neg p] \wedge [\neg q])$.]

3. (Rosen, Exercise 41, p. 43) Are the predicate statements $(\forall x)\ [P(x) \Rightarrow Q(x)]$ and $[\forall x, P(x)] \Rightarrow Q(x)$ logically equivalent? Explain why or give a counterexample.

4. (Rosen, Exercise 43, p. 43) Are the predicate statements $(\exists x)\ [P(x) \vee Q(x)]$ and $[\exists x, P(x)] \vee [\exists\ x, Q(x)]$ logically equivalent? Explain why or give a counterexample.

5. (Rosen, Exercise 46, p. 43) Show that the predicate statements

$$(\forall x)\ [P(x) \vee Q(x)] \quad \text{and} \quad [\forall x, P(x)] \vee [\forall x, Q(x)]$$

are not logically equivalent.

6. (Rosen, Exercise 47, p. 43) Show that the predicate statements

$$(\exists x)\ [P(x) \wedge Q(x)] \quad \text{and} \quad [\exists x, P(x)] \wedge [\exists x, Q(x)]$$

are not logically equivalent.

7. (Rosen, Exercise 4, p. 51) Let $P(x, y)$ be the statement, **"***if x is a student and y is a class, then x has taken class y.***"** Express the following statements in everyday language
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- **(a)** $\exists x\ \exists y\ P(x, y)$
- **(b)** $\exists x\ \forall y\ P(x, y)$
- **(c)** $\forall x\ \exists y\ P(x, y)$
- **(d)** $\exists y\ \forall x\ P(x, y)$
- **(e)** $\forall y\ \exists x\ P(x, y)$
- **(f)** $\forall x\ \forall y\ P(x, y)$

8. (Rosen, Exercise 24, p. 54) Translate each of the following mathematical statements into everyday language:
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- **(a)** $\exists x\ \forall y\ (x + y = y)$
- **(b)** $\forall x\ \forall y\ \left(x \geq 0 \wedge y < 0 \Rightarrow x - y \geq 0\right)$
- **(c)** $\forall x\ \forall y\ \left([x \neq 0] \wedge [y \neq 0] \Leftrightarrow xy \neq 0\right)$
- **(d)** $\exists x\ \exists y\ \left([x^2 \geq y] \wedge [x < y]\right)$
- **(e)** $\forall x\ \forall y\ \exists z\ (x + y = z)$
- **(f)** $\forall x\ \forall y\ \left([x < 0] \wedge [y < 0] \Leftrightarrow xy > 0\right)$

9. (Rosen, Exercise 50, p. 76) Prove that either $2 \cdot 10^{500}$ or $2 \cdot 10^{500} + 1$ is not a perfect square. Is the proof constructive (does it say that a specific choice is not a perfect square) or it nonconstructive?

10. (Rosen, Exercise 51, p. 76) Prove that there is a pair of consecutive integers such that one is a perfect square and the other is a perfect cube.

11. (Rosen, Exercise 52, p. 76) Prove that the product of two of the numbers

$$65 \cdot 10^{1000} - 8^{2001} + 3^{177}, \quad 79^{1212} - 9^{2399} + 2^{2001}, \quad 24^{4493} - 5^{8192} + 7^{1777}$$

is nonnegative without evaluating any of these numbers. Is the proof constructive or nonconstructive?

12. (Rosen, Exercise 57, p. 76) Prove that if n is an odd integer, then there is a unique integer k such that n is the sum of $k - 2$ and $k + 3$.

13. (Rosen, Exercise 11, p. 115) Let **P**(m, n) be the statement that if m and n are positive integers then m (evenly) divides n. Determine which of the following statements are true.
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- **(a)** **P**$(4, 5)$
- **(b)** **P**$(2, 4)$
- **(c)** $\forall m\ \forall n$ **P**(m, n)
- **(d)** $\exists m\ \forall n$ **P**(m, n)
- **(e)** $\exists n\ \forall m$ **P**(m, n)
- **(f)** $\forall n$ **P**$(1, n)$

14. (Rosen, Exercise 26, p. 116) Let x be a real number. Prove that x^3 is an irrational real number, then x is also irrational.

15. (Rosen, Exercise 30, p. 116) Prove that there is a positive integer which can be written as a sum of squares of two other positive integers in more than one way. **[** ***Hint:*** Start with the Pythagorean triples $3, 4, 5$ and $5, 12, 13$.**]**

16. (Rosen, Exercise 31, p. 116) Disprove the statement that every positive integer is the sum of the cubes of eight nonnegative integers. **[** ***Note:*** A result due to J. – L. Lagrange states that every positive integer is the sum of the squares of four integers.**]**

17. (Rosen, Exercise 32, p. 116) Disprove the statement that every positive integer is the sum of at most two squares and a cube of nonnegative integers.

18. (Rosen, Exercise 47, p. 225) Prove or disprove that if you have an 8 gallon jug of water and you have empty jugs with capacities of 5 and 3 gallons respectively, then you can measure four gallons of water by successively pouring the contents of one jug into another jug.

Additional exercises:

1. Suppose that **P, Q** and **R** are logical statements such that **(P ∨ R) ⇔ (Q ∨ R).** Give a counterexample to show that the latter does not necessarily imply **P ⇔ Q.** Also, give a counterexample to show that the analogous condition **(P ∧ R) ⇔ (Q ∧ R)** does not necessarily imply **P ⇔ Q.**

2. Suppose that **Q(x, y)** is the following predicate statement:

If **x** *and* **y** *are odd positive integers, then* $\mathbf{y}^{\mathbf{x}}$ *is a perfect square.*

Determine whether each of the statements $\exists x\ \forall y\ Q(x, y)$ and $\forall x\ \exists y\ Q(x, y)$ is true or false. How does the answer change if the word "odd" is removed?

> **RECALL:** A positive integer $p > 1$ is said to be *prime* if the only possible factorizations of **p** into a product **a b** of positive integers are given by $a = 1$ and $b = p$, or $b = 1$ and $a = p$.

3. Find a counterexample to show that the following conjecture is not true: Every positive integer can be expressed as a sum $p + a^2$, where $a \geq 0$ is an integer and **p** is either a prime or equal to 1.

4. Prove that the only prime number of the form $n^3 - 1$ is given by 7. **[*Hint*:** Look at the usual factorization for the difference of two cubes.**]**

5. Prove that the only prime number **p** such that $3p + 1$ is a perfect square is given by 5. **[*Hint*:** Look at the usual factorization for the difference of two squares.**]**

II.1 : Notation and first steps

(Halmos, § 1**;** Lipschutz, §§ 1.2 – 1.5, 1.10**)**

Questions to answer.

1. Give nonmathematical counterexamples to show that the following statements about set – theoretic membership are not necessarily true for arbitrary sets **A, B, C.**

- ***(i)*** $A \in A$.
- ***(ii)*** If $A \in B$, then $B \in A$.
- ***(iii)*** If $A \in B$ and $B \in C$, then $A \in C$.

2. Give a nonmathematical example of sets **A, B, C** such that $A \subset B$ and $B \in C$ but $A \notin C$.

3. In the set – theoretic approach to classical geometry, space is a set and the points are the elements of that set. Each line or plane will correspond to a subset of space. How might one interpret the concept of a line lying on a plane?

Problems for study.

Lipschutz : 1.1

II.2 : Simple examples

(Halmos, §§ 1 – 3**;** Lipschutz, § 1.12**)**

Problems for study.

Lipschutz : 1.6, 1.41, 1.43(*bcd*), 1.44

Exercises to work.

1. Suppose **A, B** and **C** are sets such that $\mathbf{A} \subset \mathbf{B} \subset \mathbf{C} \subset \mathbf{A}$. Prove that $\mathbf{A} = \mathbf{B} = \mathbf{C}$.

2. Suppose **A, B** and **C** are sets such that **A** is properly contained in **B** and **B** is properly contained in **C.** Prove that **A** is properly contained in **C.**

3. On page 10 of Halmos, the following sets are listed

$$\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}\}$$

and an exercise in the middle of the page asks if these sets are distinct. Determine whether this is true and give reasons for your answer.

4. Is it possible to find set **A** and **B** such that both $\mathbf{A} \in \mathbf{B}$ and $\mathbf{A} \subset \mathbf{B}$ are true? Give an example or prove this is impossible.