Supplement to VI.4 — Counting order types for a countably infinite set

By Exercise VI.4.6, the number of partial orderings on the set $\mathbf{N}$ of nonnegative integers is equal to $2^{\aleph_0}$, and there is an assertion that this is the same as the cardinality of the set of all order types of partial orderings on $\mathbf{N}$. The purpose here is to prove this fact.

As usual, one distinguishes order types using the properties of specific partial orderings. In fact, the examples of interest of us will all be **linear** orderings.

THE EXAMPLES. Let $A \subset \mathbf{N}$ be an arbitrary subset, and for each such subset A let S_A be given by the set

$$S \;=\; \mathbf{N} \,\cup\, \left(\bigcup_{a \in A} [2a, 2a+1] \,\cap\, \mathbf{Q} \right) .$$

Then each S_A is a subset of the rational numbers which contains the nonnegative integers $\mathbf{N}$, and as such each set S_A is countably infinite and has a linear ordering given by restricting the usual ordering of the rational numbers. We need to show that S_A and S_B have different order types if $A \neq B$. For this we need an auxiliary concept.

Definition. Let Y be a partially ordered set. A point $y_0 \in Y$ is said to be *left semi-isolated* if there is a greatest element of Y which is less than y_0, and the point y_0 is said to be *right semi-isolated* if there is a least element of Y which is greater than y_0.

If S_A is one of the sets described above, then we have the following:

> *The set of left semi-isolated points that are not right semi-isolated is the set of all points of the form $2a$ for some $a \in A$.*

Given a partially ordered set Y, let $\mathbf{SLSI}(Y)$ be the set of all $y \in Y$ that are left semi-isolated but not right semi-isolated (*i.e.*, the set of *strictly* left semi-isolated points). The following result is an immediate consequence of this observation and the definitions.

Proposition. *If U and V are partially ordered sets and $f : U \to V$ is an order-isomorphism, then f maps the set $\mathbf{SLSI}(U)$ onto the set $\mathbf{SLSI}(V)$. Similarly, f maps the set $\mathbf{SI}(U)$ of all left or right semi-isolated points in U onto the set $\mathbf{SI}(V)$ of all left or right semi-isolated points in V.*■

We can now show that S_A and S_B have different order types as follows. If $f : S_A \to S_B$ is an order-isomorphism, then f sends $\mathbf{N}$ to itself because $\mathbf{N} = \mathbf{SI}(S_A)$ and $\mathbf{N} = \mathbf{SI}(S_B)$. Furthermore, since $\mathbf{SLSI}(S_A)$ is equal to $\{n \in \mathbf{N} \mid n = 2a, \text{ some } a \in A\}$ and similarly the set $\mathbf{SLSI}(S_B)$ is equal to $\{n \in \mathbf{N} \mid n = 2b, \text{ some } b \in B\}$, it follows that a nonnegative integer n is equal to $2a$ for some $a \in A$ if and only if it is equal to $2b$ for some $b \in B$. But this means that $A = B$, and therefore we have shown that A must be equal to B if S_A and S_B have the same order type.■