I : General considerations

This is an upper level undergraduate course in set theory. There are two official texts**.**

P. R. Halmos, ***Naive Set Theory*** (Undergraduate Texts in Mathematics).
Springer – Verlag, New York, 1974. ISBN: 0–387–90092–6.

This extremely influential textbook was first published in 1960 and popularized the name for the **"**working knowledge**"** approach to set theory that most mathematicians and others have used for decades. Its contents have not been revised, but they remain almost as timely now as they were nearly fifty years ago. The exposition is simple and direct. In some instances this may make the material difficult to grasp when it is read for the first time, but the brevity of the text should ultimately allow a reader to focus on the main points and not to get distracted by potentially confusing side issues**.**

S. Lipschutz, ***Schaum's Outline of Set Theory and Related Topics***
(Second Ed.). McGraw–Hill, New York, 1998. ISBN 0–07–038159–3.

The volumes in Schaum's Outline Series are designed to be extremely detailed accounts that are written at a level accessible to a broad range of readers, and this one is no exception. As such, it stands in stark contrast to Halmos, and in this course it will serve as a workbook to complement Halmos.

The following book has also been used for this course in the past and might provide some useful additional background. It is written at a higher level than Halmos but it is also contains very substantially more detailed information**.**

D. Goldrei, ***Classic Set Theory: A guided independent study***.
Chapman and Hall, London, 1996. ISBN 0–412–60610–0.

Still further references (*e.g.*, the text for Mathematics 11 by K. Rosen) will be given later.

These ***course notes*** are designed as a further source of official information, generally at a level somewhere between the two required texts. Comments on both Halmos and Lipschutz will be inserted into these notes as they seem necessary.

I.1 : Overview of the course

(Halmos, *Preface***; Lipschutz,** *Preface***)**

Set theory has become the standard framework for expressing most mathematical statements and facts in a formal manner. Some aspects of set theory now appear at nearly every level of mathematical instruction, and words like *union* and *intersection* have become almost as standard in mathematics as *addition*, *multiplication*, *negative* and ***zero.*** The purpose of this course is to cover those portions of set theory that are used and needed at the advanced undergraduate level.

In the preface to ***Naive Set Theory*****,** P. R. Halmos (1916 –) proposes the following characterization of the set – theoretic material that is needed for specialized undergraduate courses in mathematics:

> *Every mathematician agrees that every mathematician must know some set theory; the disagreement begins in trying to decide how much is some. The purpose … is to tell the beginning student the basic set-theoretic facts … with the minimum of philosophical discourse and logical formalism. The point of view throughout is that … the concepts and methods … are merely some of the standard mathematical tools.*

Following Halmos, whose choice of a book title was strongly influenced by earlier writings of H. Weyl (1885 – 1955), mathematicians generally distinguish between the "naïve" approach to set theory which provides enough background to do a great deal of mathematics and the axiomatic approach which is carefully formulated in order to address tough questions about the logical soundness of the subject. We shall discuss some key points in the axiomatic approach to set theory, but generally the emphasis will be on the naïve approach. The following quotation from Halmos provides some basic guidelines:

> *axiomatic set theory from the naïve point of view … axiomatic in that some axioms for set theory are stated and used as the basis for all subsequent proofs … naïve in that the language and notation are those of ordinary informal (but formalizable) mathematics. A more important way in which the naïve point of view predominates is that set theory is regarded as a body of facts, of which the axioms are a brief and convenient summary*

The Halmos approach to teaching set theory has been influential and has proven itself in a half century of use, but there is one point in the preface to ***Naive Set Theory*** that requires comment:

> *In the orthodox axiomatic view [of set theory] the logical relations among various axioms are the central objects of study.*

An entirely different perspective on axiomatic set theory is presented in the following online site:

http://plato.stanford.edu/entries/set-theory

Much of the research in axiomatic set theory that is described in the online site involves (1) the uses of set theory in other areas of mathematics, and (2) testing the limits to which our current understanding of mathematics can be safely pushed.

There is some overlap between the contents of this course and the lower level course ***Mathematics 11: Discrete Mathematics*****.**   Both courses cover basic concepts and terms from set theory, but there is more emphasis in the former on counting problems and more emphasis here on abstract constructions and properties of the real number system. A related difference is that there is more emphasis on finite sets in Mathematics 11. At various points in the course it might be worthwhile to compare the treatment of topics in this course and its references with the presentation in the corresponding text for Mathematics 11:

> K. H. Rosen, ***Discrete Mathematics and Its Applications*** (Fifth Ed.). McGraw – Hill, New York, 2003. ISBN: 0– 07293033– 0. *Companion Web site:* http://www.mhhe.com/math/advmath/rosen/

Some supplementary exercises from this course will be taken from Rosen, and supplementary references to it will also be given in these notes as appropriate.

One basic goal of an introduction to the foundations of mathematics is to explain how mathematical ideas are expressed in writing. Therefore a secondary aim of these notes (and the course) is to provide an overview of modern mathematical notation. In particular, we shall attempt to include some major variants of standard notation that are currently in use.

At some points of these notes there will be discussions involving other areas of the mathematical sciences, mainly from lower level undergraduate courses like calculus (for functions of one or several variables), discrete mathematics, elementary differential equations, and elementary linear algebra. The reason for such inclusions is that we are developing a foundation for the mathematical sciences, and in order to see how well such a theory works it is sometimes necessary to see how it relates to some issues from other branches of the subject(s).

The most important justification for the course material is that provides a solid, relatively accessible logical foundation for the mathematical sciences and an overview of how one reads and writes mathematics. However, this does not explain how or why set theory was developed, and some knowledge of these points is often useful for understanding the mathematical role of set theory and the need for some discussions that might initially seem needlessly complicated. At various points in these notes – and particularly for the rest of this unit – we shall include material to provide historical perspective and other motivation.

Starred proofs and appendices

We shall follow the relatively standard notational convention and mark proofs that are more difficult, or less central to the course, by one to four stars. Generally the number of stars reflects a subjective assessment of relative difficulty or importance; items not marked with any starts have the highest priority, items with one star have the next highest priority, and so on. Section V.3 is an exception to this principle for the reasons given at the beginning of that portion of the notes.

There are also several appendices to sections in the notes; these fill in mathematical details or cover material that is not actually part of the course but is closely related and still worth knowing. Since this material can be skipped without a loss of logical continuity, we have also passed on inserting stars in the appendices.

Exercises

As in virtually every mathematics course, working problems or exercises is important, and for each unit there are lists of questions, problems or exercises to study or attempt. Normally the exercises for a section will begin with a list of examples from Lipschutz called ***"Problems for study."*** Solutions for all these are given in Lipschutz, but attempting at least some of them before looking at the solutions is strongly recommended. Each section will also have a list of ***"Questions to answer"*** or ***"Exercises to work."*** Answers and solutions for these will be given separately.

I.2 : Historical background and motivation

It is important to recognize that mathematicians did not develop set theory simply for pedagogical or aesthetic reasons, but on the contrary they did so in order to understand specific problems in some fundamentally important areas of the subject. Three of the most important influences in the development of set theory were the following

1. There was an increasing awareness among later 19th century mathematicians that a more secure logical framework for mathematics was needed.

2. Several 19th century mathematicians and logicians discovered the algebraic nature of some basic rules for deductive logic.

3. Most immediately, there was a great deal of research at the time to understand the representations of functions by means of trigonometric series.

The first of these reflects the *unavoidable need* for something like set theory in modern mathematics, while the second reflects the *formal structure* of set theory and the third reflects its *principal substance*, which is *the study of sets that are infinitely large*. In brief, these are the *"why,"* the *"how,"* and the *"what"* of set theory. We shall discuss each of these in the order listed.

At various points in this section and elsewhere in these notes, we shall refer to the text for the course ***Mathematics 153: History of Mathematics*****:**

> D. M. Burton, ***The History of Mathematics, An Introduction*** (Sixth Ed.). McGraw – Hill, New York, 2006. ISBN: 0– 073– 05189– 6.

The excellent online ***MacTutor History of Mathematics Archive*** located at the site http://www-groups.dcs.st-and.ac.uk/~history/index.html

contains extensive biographical information for more than 1100 mathematicians (including many women and individuals from non-Western cultures) as well as an enormous amount of other material related to the history of mathematics.

We now begin our summary of historical influences leading to the development of set theory.

The need for more reliable logical foundations. Most areas of human knowledge are now organized using deductive logic in some fashion, and the ancient Greek formulation of mathematics in such terms was one of the earliest and most systematic examples. With the discovery of irrational numbers, Greek mathematics used geometrical ideas as their logical foundation for mathematics, and with the passage of time Euclid's ***Elements*** emerged as the standard reference. This standard for logical soundness remained unchanged for nearly 2000 years, and the following quotation from the works of Isaac Barrow (1630 – 1677) reflects this viewpoint:

> Geometry is the basic mathematical science, for it includes arithmetic, and mathematical numbers are simply the signs of geometrical magnitude.

Barrow's viewpoint was adopted in the celebrated work, *Philosophiæ Naturalis Principia Mathematica*, written by his student Isaac Newton (1642 – 1727). On the other hand, the development of calculus in the 17^{th} century required several constructions that did not fit easily into the classical Greek setting. In this context, it is slightly ironic that Barrow deserves priority for several important discoveries leading to calculus.

A simple – probably much too simple – description of calculus is that it is a set of techniques for working with quantities that are limits of successive approximations. Probably the simplest illustration of this is the area of a circle, which is the limit of the areas of regular **n** – sided polygons that are inscribed within, or circumscribed about, the circle as **n** becomes increasingly large. During the Fifth Century B. C. E., Greek mathematicians and philosophers discovered that a casual approach to infinite processes could quickly lead to nontrivial logical difficulties; the best known of these are contained in several well known paradoxes due to Zeno of Elea (*c*. 490 – 425 B. C. E.; see pages 103 – 104 of Burton for more details). The writings of Aristotle (384 – 322 B. C. E.) in the next century helped set a course for Greek mathematics that avoided the **"**horror of the infinite.**"** When Archimedes (287 – 212 B. C. E.) solved numerous problems from integral calculus, his logically rigorous proofs of the solutions used elaborate arguments by contradiction in which he studiously avoided questions about limits.

This stiff resistance to thinking about the infinite eventually weakened, in part due to influences from Indian mathematics, which was far more open to discussing infinity, and also in part due various investigations in mathematics and philosophy during the late Middle Ages. When interest in problems from calculus reappeared towards the end of the 16^{th} century, there were many workers in the area who used infinite processes freely, while there were also some who had reservations about some or all such techniques. Since the methods of calculus were giving reliable and consistent answers to questions that had been previously out of reach, the resolution of such misgivings was an important issue. In the discussions of this problem which took place during the 17^{th} and

18th centuries, it had become clear that calculus involves limit concepts that are beyond normal geometrical experience. We shall not attempt to retrace the entire development of this, but instead we shall concentrate on some important developments from the 19th century. The first of these was the relatively precise definition of limit due to A. – L. Cauchy (1789 – 1857) in 1821; this was further refined into the modern definition of limit using δ and ε which is due to K. Weierstrass (1815 – 1897). Another important development was the critical analysis of convergence questions for infinite series, particularly in the writings of N. H. Abel (1802 – 1831). A third development was the realization that certain basic facts about continuous functions required rigorous logical proofs. Examples include the Intermediate Value Theorem and its proof by B. Bolzano (1781 – 1848). This listing of developments is definitely (and deliberately!) not exhaustive, but it does illustrate the 19th century activity to put the content of calculus on a logically sound foundation.

Ultimately such basic facts from calculus depend upon a firm understanding of the real numbers themselves. Greek mathematicians turned to geometry as a foundation for mathematics precisely because their understanding of the real numbers was incomplete. However, the work of Eudoxus of Cnidus (*c*. 408 – 355 B. C. E.) yielded one very important property of real numbers; namely, between any two real numbers there is a rational number. By the end of the 16th century our usual understanding of real numbers in terms of infinite decimals was a well established principle in European mathematics, science and engineering. The final insight in the process was due to R. Dedekind (1831 – 1916), and it was a converse to the principle implicitly due to Eudoxus; specifically, the real numbers are in some sense the ***largest possible number system*** in which everything can be approximated by rational number to any desired degree of accuracy. ***Justifying this viewpoint in a logically rigorous manner requires the methods and results of set theory.***

At the same time that mathematicians were developing a new logical foundation for calculus during the 18th and 19th centuries, still other advances in mathematics led to even more serious questions about the foundations of mathematics as they had been previously understood. One philosophical basis for using geometry as a foundation for mathematics is to view the postulates of Euclidean geometry as absolutely inevitable necessities of thought, much like the fact that two plus two equals four. In particular, the 18th century philosophical writings of I. Kant (1724 – 1804) were particularly influential in viewing the basic facts of geometry as intuitions that are independent of experience. When 19th century mathematicians such as J. Bolyai (1802 – 1860), N. Lobachevsky (1793 – 1856) and C. F. Gauss (1777 – 1855) realized that there was a logically consistent alternative to the axioms for Euclidean geometry, the Kantian position became far more difficult to defend. Further information on the Non – Euclidean geometry studied by these mathematicians appears on pages 561 – 601 of Burton.

The development of a mathematically rigorous treatment of calculus had an implication for classical Euclidean geometry that was largely unanticipated. When mathematicians examined classical geometry in light of the logical standards that they needed for calculus, they realized that the classical framework did not meet the new standards. For example, concepts like betweenness of points on a line and points lying on the same or different sides of a line were generally ignored in Euclid's ***Elements***. One way to illustrate the need for treating such matters carefully is to see what can go wrong if they are dismissed too casually. A standard example in this direction is the "proof" in the online reference below, which is attributed to W. Rouse Ball (1850 – 1925). This looks very much like a classical Greek proof, but it reaches the obviously false conclusion that every triangle is isosceles:

http://www.mathpages.com/home/kmath392.htm

The need to repair the foundations of classical Greek geometry further underscored the urgent need to have an entirely new logical foundation for mathematics.

In fact, the adoption of set theory as a foundation for mathematics is also a key step towards bringing classical Greek geometry up to modern logical standards. A discussion of this work is beyond to scope of these notes, but some further information is contained on pages 619 – 621 of Burton.

The use of algebraic methods to analyze logical questions. Traditionally, logic was studied as a branch of philosophy, and the ancient Greek approach to mathematics established the role and usefulness of logic in studying mathematics. Eventually mathematicians and logicians realized that, conversely, some ideas from mathematics were also useful in the analysis of logic. Some early examples of logical symbolism appear in the work of J. L. Vives (1492 – 1540) and J. H. Alsted (1588 – 1638). Fairly extended discussions appear in papers of G. W. von Leibniz (1646 – 1716) that were not published during his lifetime, and during the 18^{th} century there were several further tentative probes in this direction by others such as Ch. von Wolff (1679 – 1754), G. Ploucquet (1716 – 1790), J. H. Lambert (1728 – 1777), and L. Euler (1707 – 1783). However, sustained and productive interest in the mathematical aspects of logic began in the middle of the 19^{th} century, and since that time mathematical ideas have played a very important (but not exclusive) role in this subject. More recently, the importance of formal logic for computer science has been a major source of motivation for further research.

The name *mathematical logic* is due to G. Peano (1858 – 1932), and the subject is also often called symbolic logic (although not everyone necessarily agrees these terms have identical meanings). Mathematical logic still includes the logic of classical civilizations, for example as summarized in the ***Organon*** of Aristotle or the ***Nyaya Sutras*** of the Indian Philosopher Aksapada Gautama (conjecturally around the Second Century B. C. E., but possibly as early as 550 B. C. E. or as late as 150 A. D.), or the logic that was developed in ancient Chinese civilization probably around the time of Aristotle, but it is developed more like a branch of abstract algebra.

The emergence of mathematical methods as an important factor in logic was firmly established with the appearance of the book, ***The Mathematical Analysis of Logic***, by G. Boole (1815 – 1864) in 1847. Boole's work contained a great deal of new material, but in some respects it also drew upon earlier discoveries, writings and ideas due to R. Whately (1787 – 1863), G. Peacock (1791 – 1858), G. Bentham (1800 – 1884, better known for his work as a botanist), A. De Morgan (1806 – 1871) and William Stirling Hamilton (1788 – 1856); it should be noted that the latter was a Scottish logician and not the same person as the better known Irish mathematician William Rowan Hamilton (1805 – 1865), who is recognized for several fundamental contributions to mathematics, including his mathematical approach to classical physics and the invention of quaternions. The following is a typical example of a conclusion that followed from the methods of these 19^{th} century logicians but not from classical Aristotelian logic:

- In a particular group of people,
  - most people have shirts
  - most people have shoes
  - **therefore,** some people have both shirts and shoes.

Other contributors during the second half of the 19^{th} century included J. Venn (1834 – 1923), who devised the pictorial representations of sets that now carry his name, and C. L. Dodgson (1832 – 1898), who is better known by his literary pseudonym *Lewis Carroll.* His interests covered a very broad range of topics, and his mathematical achievements include some deep studies in symbolic logic and logical reasoning. Much of this work involved specific logical problems of a somewhat whimsical nature, but he also made some noteworthy contributions in more general directions, including the use of truth tables. All this activity in logic led to fairly definitive algebraic formulations by W. S. Jevons (1835 – 1882) and E. Schröder (1841 – 1902).

Further discussion of the work of Boole and De Morgan (as well as other topics that are mentioned above) appears on pages 643 – 647 of Burton.

Representations of functions by trigonometric series. Several distinct areas in mathematical physics – most notably, wave motion and heat flow – motivated interest in expressing periodic functions satisfying $\mathbf{f(x + 2\pi) = f(x)}$ by means of an infinite series of trigonometric functions

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\ nx + b_n \sin nx)$$

analogous to the power series expansions of the form

$$\sum_{i=0}^{\infty} a_i z^i,$$

that are so useful for many purposes. A discussion of such series at the level of first year calculus appears in Sections 8.9 and 8.10 of the following classic calculus text:

> R. L. Finney, M. D. Weir, F. R. Giordano, ***Thomas' Calculus, Early Transcendentals*** (Tenth Ed.). Addison – Wesley, Boston, 2000. ISBN: 0–201–44141–1.

During the middle of the 19^{th} century many prominent mathematicians studied aspects of the following question:

> *To what extent is the representation of a function by a (possibly infinite) trigonometric series unique?*

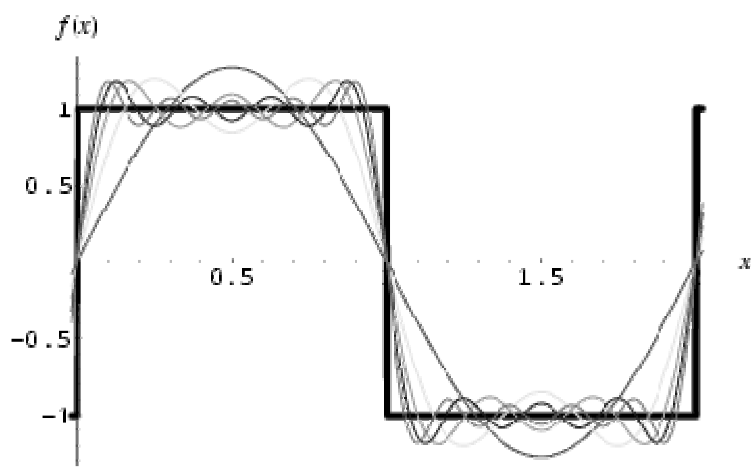
The founder of set theory, **Georg Cantor** (1845 – 1918), gave a positive answer to this question in 1870.

Theorem. *Suppose that we are given two expansions of a reasonable function* **f** *as a convergent trigonometric series:*

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \; = \tfrac{1}{2}a_0' + \sum_{n=1}^{\infty} a_n' \cos(nx) + \sum_{n=1}^{\infty} b_n' \sin(nx)$$

Then $\mathbf{a_n = a_n'}$ *and* $\mathbf{b_n = b_n'}$ *for all nonnegative integers* **n.**

This is a pretty good conclusion, but one actually would like a little more. We have not specified what we mean by a reasonable function, and indeed we should like to include some functions that are not necessarily continuous. The most basic example in this context is the so – called ***square wave function*** whose value from 0 to π is + 1 and whose value from π to 2π is – 1. Waves of this type occur naturally in several physical contexts: The graph of the square wave function (with the **x –** axis rescaled in units of π) is given below.



(SOURCE: http://mathworld.wolfram.com/FourierSeriesSquareWave.html)

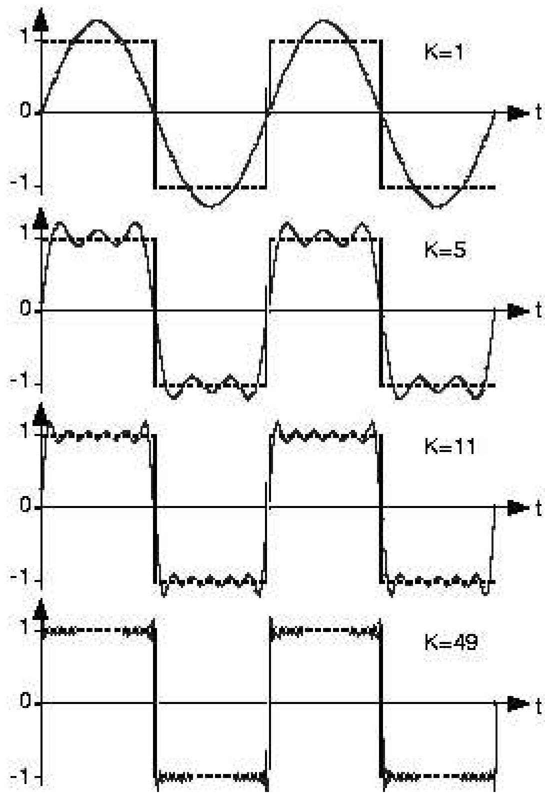
Obviously this function is discontinuous, with a jump in values at every integral multiple of π, and one might suspect that it really does not matter how we might define the function at such sparsely distributed jump discontinuities. In fact, this is the case, and for every such choice one obtains the same trigonometric series representing the square wave function:

$$f(x) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right).$$

(This is the general expression for period 2**L,** so here $\mathbf{L = \pi}$.)

Here are some graphs to show how close the partial sums come to approximating the square wave. Note that the graphs suggest the value of the infinite sum is zero at integral multiples of π (this is in fact true, but we shall not go into the details). Here is a reference for this illustration.

http://cnx.rice.edu/content/m0041/latest/



Clearly we could carry out the same construction for higher frequency square waves (using positive integral multiples of 2π) and find examples of reasonable functions with the same trigonometric series such that the values of the functions are the same except for some arbitrarily large finite set of values between 0 and 2π. This leads naturally to the following problem that Cantor considered in connection with his basic uniqueness result:

Do two reasonable functions have the same Fourier series if they agree at all but an infinite sequence of points $\mathbf{p_n}$ *between* 0 *and* 2π?

Cantor showed that the answer was <u>*yes*</u> if the sequence had the following ***closure property:*** *If a subsequence* $\mathbf{p_{n(k)}}$ *converges to a limit* **L,** *then* $\mathbf{L = p_m}$ *for some* **m.**

Subsequent work established the result without the closure hypothesis. Further information on these matters may be found in the following reference (which is definitely ***not*** written at the advanced undergraduate level – the citation is included for the sake of completeness):

> A. S. Kechris and A. Louveau, ***Descriptive set theory and the structure of sets of uniqueness*** (London Math. Soc. Lect. Notes Vol. 128). Cambridge University Press, Cambridge, UK, and New York, 1987. ISBN: 0–521–35811–6.

The important point of all this for our purposes is that Cantor's analysis of the exceptional points led him to abstract set – theoretic concepts and ultimately to his extremely original (and at first highly controversial) research on set theory. Additional information on Cantor and his work appears on pages 668 – 690 of Burton. Further developments in the history of set theory are discussed on pages 690 – 707 of Burton, but the material covered after the middle of page 701 is not discussed in this course.

Some further references

Additional historical background on the topics discussed in this section is given in the following online sites.

http://math.ucr.edu/~res/math153/history03.pdf

This site discusses some issues related to the logical gaps in Euclid's ***Elements*** and why the latter should be still be viewed very positively despite such problems.

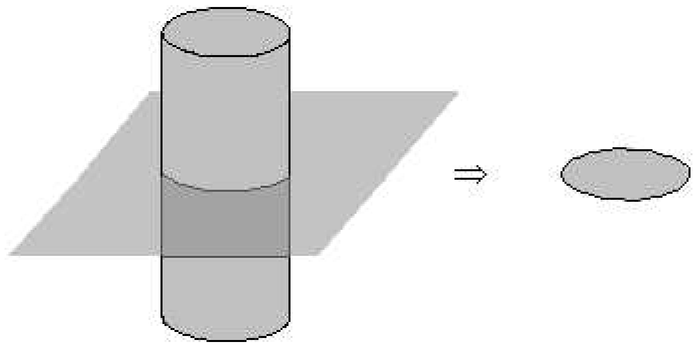
http://math.ucr.edu/~res/math153/history12.pdf

http://math.ucr.edu/~res/math153/history14a.pdf

The first document contains an account of infinitesimals which goes beyond the Appendix to this section in some respects, and it also includes further discussion on problems with the logical soundness of calculus that arose during the period from 1600 to 1900. The second document describes one noteworthy example to illustrate how an overly casual approach to manipulating infinite series can lead to fallacious conclusions.

I.2. Appendix : Comments on infinitesimals

One of the major logical problems with calculus as developed in the 17th century was the legitimacy of objects called ***infinitesimals***. The idea is well illustrated in the method employed by B. Cavalieri (1598 – 1647) to study the volume of a solid **A** that is contained between two parallel planes. If the planes are defined by the equations $\mathbf{z} = 0$ and $\mathbf{z} = 1$, then for each **t** between 0 and 1 one has the cross section $\mathbf{A_t}$ formed by Intersecting **A** with the parallel plane defined by $\mathbf{z} = \mathbf{t}$. Cavalieri's idea is to view **A** as composed of an infinite collection of cylindrical solids whose bases are the cross sections $\mathbf{A_t}$ and whose heights are some very small, in fact ***infinitesimally small,*** value that we shall call **dt.**



(FIGURE SOURCE: http://www.mathleague.com/help/geometry/3space.htm **)**

From this viewpoint, the total volume is obtained by adding the volumes of these infinitesimally short cylindrical solids; in modern terminology, one adds or integrates these infinitesimals by taking the definite integral of the area function with respect to **t** from 0 to 1. Of course, the point of this discussion is to convince the reader that the volume of **A** is given by the following standard integral formula in which **a(t)** denotes the area of the planar section $\mathbf{A_t}$**:**

$$V = \int_0^1 a(t)\,dt$$

This is an excellent heuristic argument, but its logical soundness depends upon describing the concept of an infinitesimal precisely. It was clear to 17^{th} and 18^{th} century scientists and philosophers that such infinitesimals were supposed to be smaller than any finite quantity but were still supposed to be positive. If one is careless with such a notion it is easy to contradict the principle that between any two real numbers there is a rational number; a crucial question is *whether it is ever possible to be careful enough to avoid these or other logical difficulties.* Although proponents of calculus made vigorous efforts to explain infinitesimals and were getting reliable answers, their explanations did not really clarify the situation much to mathematicians or others of that era. A clear and rigorous foundation for calculus was not achieved until infinitesimals were discarded (for foundational purposes) in the 19^{th} century and the subject was based upon the concept of limit (see the discussion above).

Despite their doubtful logical status, many users of mathematics have continued to work with infinitesimals, probably motivated by their relative simplicity, the fact that they gave reliable answers, and an expectation that mathematicians could ultimately find a logical justification for whatever was being attempted. This attitude towards infinitesimals was also evident in many undergraduate textbooks in mathematics, science and engineering, particularly through the first half of the 20^{th} century; the following is a typical example:

> W. A. Granville, P. F. Smith and W. R. Longley, ***Elements of Differential and Integral Calculus*** (Various editions from 1904 to 1962). Wiley, New York, 1962. ISBN: 0– 471– 00206– 2.

During the nineteen sixties Abraham Robinson (1918 – 1974) used extensive machinery from set theory and abstract mathematical logic to prove that one can in fact construct a number system with infinitesimals that satisfy the expected formal rules.   However, the crucial advantage of Robinson's concept of infinitesimal — its logical soundness — is balanced by the fact that, unlike 17^{th} century infinitesimals, it is neither simple nor intuitively easy to understand. The associated theory of ***Nonstandard Analysis*** has been studied to a considerable extent mathematically, but it is not widely used in the traditional applications of the subject to the sciences and engineering; on the other hand, some recent work in mathematical economics has been formulated within the context of nonstandard analysis. The following online references provide further information on this subject:

http://members.tripod.com/PhilipApps/nonstandard.html

http://www.haverford.edu/math/wdavidon/NonStd.html

http://mathforum.org/dr.math/faq/analysis_hyperreals.html

http://en.wikipedia.org/wiki/Nonstandard_analysis

http://www.math.uiuc.edu/~henson/papers/basics.pdf

Here are a few textbook references for nonstandard analysis:

J. M. Henle and E. M. Kleinberg, ***Infinitesimal Calculus.*** Dover Publications, New York, 2003. ISBN: 0– 486– 42886– 9.

J. L. Bell, ***A Primer of Infinitesimal Analysis*****.** Cambridge University Press, New York, 1998. ISBN: 0– 521– 62401– 0.

A. E. Hurd and P. A. Loeb, ***An Introduction to Nonstandard Real Analysis*** (Pure and Applied Mathematics, Vol. 118)***.*** Academic Press, Orlando, FL, 1965. ISBN: 0– 123– 62440– 1.

Comment on "differential" notation

In older mathematics texts and also some newer books in other subjects, expressions like **dx**, **dy** and **df** refer to infinitesimals. However, in newer mathematics books, for example the multivariable calculus text

J. E. Marsden and A. Tromba, ***Vector Calculus*** (Fifth Ed.). Freeman, New York, 2003. ISBN: 0–716–74992–0

such symbols generally have a much different meaning, and it is important to recognize this. A precise description of the current usage is beyond the scope of this course; one general suggestion is to check a textbook carefully if it contains expressions like **dx** and **dy** standing by themselves and not part of a larger expression for a derivative or an integral. This applies particularly to any mathematics book beyond first year calculus with a first edition date after 1950.

Logical rigor and modern mathematical physics

The development of nonstandard analysis during the second half of the 20th century is definitely not the final step to putting everything related to mathematics on a logically sound basis; in fact, one would expect that advances in the other sciences – particularly in physics – are likely to continue yielding new ideas on how our mathematical concepts might be stretched to deal effectively with new classes of problems. Probably the most important subject currently requiring a mathematically rigorous description is the formalism introduced by the renowned physicist R. P. Feynman (1918 – 1988) about 60 years ago to study questions in quantum electrodynamics. The value and effectiveness of Feynman's techniques in physics — and even in some highly theoretical areas of mathematics — are very widely recognized, but currently there is no general method to provide rigorous mathematical justifications for the results predicted by Feynman's machinery (however, it is possible to do so in a wide range of special cases). A comprehensive account of the mathematical aspects of Feynman's ideas is given in the book cited below, and the accompanying online references provide quick surveys of Feynman's life and work:

> G. W. Johnson and M. L. Lapidus, ***The Feynman Integral and Feynman's Operational Calculus*** (Oxford Mathematical Monographs, Corrected Ed.). Oxford University Press, Oxford, UK, and New York, 2002. ISBN: 0–19–851572–3.

http://en.wikipedia.org/wiki/Richard_Feynman

http://www.feynman.com/

http://www2.slac.stanford.edu/vvc/theory/feynman.html

I.3 : Selected problems

We shall begin with an online quotation from the site

http://en.wikipedia.org/wiki/Adjoint_functor

on introducing abstract concepts.

> *Concepts are judged according to their use in solving problems, at least as much for their use in building theories.*

Here is a more focused version of the quotation:

> ***Ideally, an abstract mathematical construction such as set theory should answer, or at least shed useful new light, on some problem(s) of recognized importance.***

Motivated by the preceding comments, we shall list a few mathematical questions of varying importance and difficulty as test cases for the usefulness of set theory.

1. Providing a clear and simple mathematical description of both relations and functions.
2. Rigorously justifying the so – called ***pigeonhole principle*** **:** If we are given **m** objects and **n** locations to put them with **m > n,** then at least one of the locations will contain at least two objects.
3. Finding a mathematically efficient and logically sound description of the real number system.
4. Understanding the likelihood that a real number which is "chosen at random" will be ***algebraic;*** *i.e.*, it is the root of a nonconstant polynomial equation with integral coefficients.

Given the fundamental importance of the real number system to analysis, it should be apparent that ***anything which will make the latter logically rigorous will play a key role in the foundations of mathematics.***

At this point a few additional remarks about the desired formulation of the real number system seem appropriate. Even though we view real numbers in terms of their infinite decimal expansions, we do not want our mathematical description of real numbers to be phrased in such terms. There are two reasons for this. One is that verifying algebraic identities for infinite decimal expansions is at best awkward; for example, consider the practical and theoretical difficulties in writing out the reciprocal to an infinite decimal expansion between 0 and 1 or writing out the positive square root of such a number. A second reason is that we would like our concept of real number to be independent of any choice of computational base, and in particular we would like a system that does not change if we replace base 10 by, say, base 2 (or 8, or 12, or 16, or 60, or …).

In an appendix to the final section of these notes we shall also consider one further question that arises naturally in connection with the points covered in this unit; namely, formulating repaired versions of classical Greek deductive geometry in terms of modern set theory.