III : Elementary constructions on sets

In this unit we cover the some fundamental constructions of set theory that are used throughout the mathematical sciences.

Much of this material is probably extremely familiar, but we shall start at the beginning for several reasons, including the following:

1. To ensure that the discussion is complete.
2. To emphasize the more abstract perspective on the material.
3. To state some subtle but important differences in terminology between these notes and more elementary treatments of the material.

In the final section of this unit we shall indicate how one expresses everything in more formal and axiomatic terms.

Numbering conventions. In mathematics it is often necessary to use results that were previously established. Throughout these notes we shall refer to results from earlier sections by notation like Proposition **II.4.6,** which will denote Proposition 6 from Section **II.4** (this particular example does not actually exist, but it should illustrate the key points adequately).

III.1 : Boolean operations

(Halmos, §§ 4 – 5; Lipschutz, §§ 1.6 – 1.7)

We shall begin with a discussion of unions, intersections and complements. In order to keep the discussion simple and familiar at the beginning, we shall begin by considering only those sets which are subsets of some fixed set **S.**

Definitions. Let **A** and **B** be subsets of some set **S.** The standard ***Boolean operations*** on these sets are defined as follows:

- The ***intersection*** of **A** and **B** is the set of all elements common to both sets. It is symbolized by $\mathbf{A} \cap \mathbf{B}$ or $\{\, \mathbf{x} \in \mathbf{S} \mid \mathbf{x} \in \mathbf{A} \text{ and } \mathbf{a} \in \mathbf{B} \,\}$.
- The ***union*** of two sets **A** and **B** is the set of elements which are in **A** or **B** or both. It is symbolized by $\mathbf{A} \cup \mathbf{B}$ or $\{\, \mathbf{x} \in \mathbf{S} \mid \mathbf{x} \in \mathbf{A} \text{ or } \mathbf{x} \in \mathbf{B} \,\}$.
- The ***relative complement*** of **A** in **S** is the set of all elements in S that do not belong to **A.** It is symbolized by $\mathbf{S} - \mathbf{A}$ or $\{\, \mathbf{x} \in \mathbf{S} \mid \mathbf{x} \notin \mathbf{A} \,\}$.

Numerous other symbols are also used for the relative complement of **A,** including

$\complement A$

and **A′** and **A^c^** and also by an **A** with a long horizontal line over it (**Ā**).

We shall now review and prove the standard relationships between these three operations on subsets of **S.** The first group describes the class of algebraic identities involving unions and intersections which appear in the writings of G. Boole.

Theorem 1. *Let* **A, B** *and* **C** *be subsets of some fixed set* **S.** *Then the union and intersection defined as above satisfy the following* ***Boolean algebra*** *identities:*

(Idempotent Law for unions.) $A \cup A = A$.

(Idempotent Law for intersections.) $A \cap A = A$.

(Commutative Law for unions.) $A \cup B = B \cup A$.

(Commutative Law for intersections.) $A \cap B = B \cap A$.

(Associative Law for unions.) $A \cup (B \cup C) = (A \cup B) \cup C$.

(Associative Law for intersections.) $A \cap (B \cap C) = (A \cap B) \cap C$.

(Distributive Law 1.) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(Distributive Law 2.) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(Zero Law.) $A \cup \emptyset = A$.

(Unit Law.) $A \cap S = A$.

The second group of relations also involves complementation.

Theorem 2. *Let* **A** *and* **B** *be subsets of some fixed set* **S.** *Then the union, intersection and relative complement satisfy the following identities:*

(Double negative Law.) $(A')' = A$.

(Complementation Law 1.) $A \cup A' = S$.

(Complementation Law 2.) $A \cap A' = \emptyset$.

(De Morgan's Law 1.) $(A \cup B)' = A' \cap B'$.

(De Morgan's Law 2.) $(A \cap B)' = A' \cup B'$.

Most if not all the verifications of these rules are fairly straightforward, and they essentially follow from the formulas for propositional calculus listed in Section II.0. We shall fill in the details atfter the next heading. Some ideitities are more obvious than others (in particular, the distributive laws and De Morgan's laws are probably less intuitive than the commutative and associative laws), and in these cases we shall also give alternate arguments that are more detailed

Boolean operations and subsets

There are simple but important characterizations of the relationship $\mathbf{A} \subset \mathbf{B}$ in terms of unions and intersections.

Theorem 3. *Let* **A** *and* **B** *be subsets of some fixed set* **S.** *Then the following are equivalent:*

(i) $\mathbf{A} \cup \mathbf{B} = \mathbf{B}$

(ii) $\mathbf{A} \subset \mathbf{B}$

(iii) $\mathbf{A} \cap \mathbf{B} = \mathbf{A}$

Proof. There are four parts to the argument.

(i) ⇒ (ii) If $\mathbf{x} \in \mathbf{A}$, then $\mathbf{x} \in \mathbf{A}$ or $\mathbf{x} \in \mathbf{B}$, and hence $\mathbf{x} \in \mathbf{A} \cup \mathbf{B}$, which is **B.** Hence we have $\mathbf{A} \subset \mathbf{B}$.

(ii) ⇒ (i) If $\mathbf{A} \subset \mathbf{B}$ and $\mathbf{x} \in \mathbf{A} \cup \mathbf{B}$, then $\mathbf{x} \in \mathbf{A}$ or $\mathbf{x} \in \mathbf{B}$, and in either case we have $\mathbf{x} \in \mathbf{B}$. Hence we have $\mathbf{A} \cup \mathbf{B} \subset \mathbf{B}$. Conversely, if $\mathbf{x} \in \mathbf{B}$, then we must have $\mathbf{x} \in \mathbf{A}$ or $\mathbf{x} \in \mathbf{B}$, so that $\mathbf{B} \subset \mathbf{A} \cup \mathbf{B}$. Combining these, we have $\mathbf{A} \cup \mathbf{B} = \mathbf{B}$.

(ii) ⇒ (iii) If $\mathbf{A} \subset \mathbf{B}$ and $\mathbf{x} \in \mathbf{A}$, then $\mathbf{x} \in \mathbf{B}$ and hence $\mathbf{x} \in \mathbf{A} \cap \mathbf{B}$, so that we have $\mathbf{A} \subset \mathbf{A} \cap \mathbf{B}$. Conversely, if $\mathbf{x} \in \mathbf{A} \cap \mathbf{B}$, then $\mathbf{x} \in \mathbf{A}$ and $\mathbf{x} \in \mathbf{B}$, and the latter means that $\mathbf{A} \cap \mathbf{B} \subset \mathbf{A}$. Combining these, we have $\mathbf{A} \cap \mathbf{B} = \mathbf{A}$.

(iii) ⇒ (ii) If $\mathbf{x} \in \mathbf{A}$, then $\mathbf{A} \cap \mathbf{B} = \mathbf{A}$ implies that $\mathbf{x} \in \mathbf{A}$ and $\mathbf{x} \in \mathbf{B}$, and the second of these means we must have $\mathbf{A} \subset \mathbf{B}$.■

Verifications of the standard identities

We shall derive the identities of Theorems 1 and 2 roughly in the order they were stated.

Idempotent laws. The law for unions is true because $\mathbf{x} \in \mathbf{A} \Leftrightarrow \mathbf{x} \in \mathbf{A}$ or $\mathbf{x} \in \mathbf{A}$, while the law for intersections is true because $\mathbf{x} \in \mathbf{A} \Leftrightarrow \mathbf{x} \in \mathbf{A}$ and $\mathbf{x} \in \mathbf{A}$.

Commutative laws. The law for unions is true because

$$\mathbf{x} \in \mathbf{A} \cup \mathbf{B} \iff \mathbf{x} \in \mathbf{A} \text{ or } \mathbf{x} \in \mathbf{B} \iff \mathbf{x} \in \mathbf{B} \text{ or } \mathbf{x} \in \mathbf{A} \iff \mathbf{x} \in \mathbf{B} \cup \mathbf{A},$$

while the law for intersections is true because

$$\mathbf{x} \in \mathbf{A} \cap \mathbf{B} \iff \mathbf{x} \in \mathbf{A} \text{ and } \mathbf{x} \in \mathbf{B} \iff \mathbf{x} \in \mathbf{B} \text{ and } \mathbf{x} \in \mathbf{A} \iff \mathbf{x} \in \mathbf{B} \cap \mathbf{A}.$$

In symbolic terms, the preceding arguments are just special cases of the more general propositional equivalences

$$\mathbf{P} \vee \mathbf{Q} \iff \mathbf{Q} \vee \mathbf{P} \quad \text{and} \quad \mathbf{P} \wedge \mathbf{Q} \iff \mathbf{Q} \wedge \mathbf{P}$$

and we shall use other such equivalences freely in deriving the remaining assertions in the theorem.

Associative laws. The argument is similar, depending upon the general propositional equivalences

$$[\mathbf{P} \wedge (\mathbf{Q} \wedge \mathbf{R})] \iff [(\mathbf{P} \wedge \mathbf{Q}) \wedge \mathbf{R}] \quad \text{and} \quad [\mathbf{P} \vee (\mathbf{Q} \vee \mathbf{R})] \iff [(\mathbf{P} \vee \mathbf{Q}) \vee \mathbf{R}]$$

where **P, Q** and **R** are the statements $\mathbf{x} \in \mathbf{A}$, $\mathbf{x} \in \mathbf{B}$ and $\mathbf{x} \in \mathbf{C}$ respectively.

Distributive laws. The argument is again similar, depending upon the general propositional equivalences

$$[\mathbf{P} \wedge (\mathbf{Q} \vee \mathbf{R})] \iff [(\mathbf{P} \wedge \mathbf{Q}) \vee (\mathbf{P} \wedge \mathbf{R})] \quad \text{and} \quad [\mathbf{P} \vee (\mathbf{Q} \wedge \mathbf{R})] \iff [(\mathbf{P} \vee \mathbf{Q}) \wedge (\mathbf{P} \vee \mathbf{R})]$$

where **P, Q** and **R** are the statements $\mathbf{x} \in \mathbf{A}$, $\mathbf{x} \in \mathbf{B}$ and $\mathbf{x} \in \mathbf{C}$ respectively.

Zero law. One can characterize the empty set as the set of all **x** such that $\mathbf{x} \neq \mathbf{x}$. Thus we have $\mathbf{x} \in \mathbf{A} \cup \emptyset \iff \mathbf{x} \in \mathbf{A}$ or $\mathbf{x} \neq \mathbf{x}$, and since the statement $\mathbf{x} \neq \mathbf{x}$ is always false the second condition is equivalent to $\mathbf{x} \in \mathbf{A}$.

Unit law. By hypothesis we know that **A** is a subset of **S** and therefore if $\mathbf{x} \in \mathbf{A}$ we also have $\mathbf{x} \in \mathbf{S}$, so that $\mathbf{x} \in \mathbf{A} \cap \mathbf{S}$. Conversely, if $\mathbf{x} \in \mathbf{A} \cap \mathbf{S}$ then we automatically have $\mathbf{x} \in \mathbf{A}$.

AN ALTERNATE APPROACH TO THE DISTRIBUTIVE LAWS. Here is a method for deriving the distributive laws that does not use abstract propositional equivalences. We include it because the equivalences in this case may be less transparent than the previous ones. Given $\mathbf{x} \in \mathbf{S}$, we know that *each one of the three fundamental statements* $\mathbf{x} \in \mathbf{A}$, $\mathbf{x} \in \mathbf{B}$ *and* $\mathbf{x} \in \mathbf{C}$ *is either true or false.* Thus there are exactly eight possibilities for every element of **S.** It will suffice to show that in each of these cases that if $\mathbf{x} \in \mathbf{A} \cap (\mathbf{B} \cup \mathbf{C})$ then $\mathbf{x} \in (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$ and conversely (this will prove the first distributive law), and similarly if we have $\mathbf{x} \in \mathbf{A} \cup (\mathbf{B} \cap \mathbf{C})$ then $\mathbf{x} \in (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$ and conversely (this will prove the second distributive law). The first step is to compile a table containing all eight possibilities; in the table below, **+** indicates that the relevant statement is true and **0** indicates that it is false.

x ∈ A	**x ∈ B**	**x ∈ C**
0	**0**	**0**
0	**0**	+
0	+	**0**
0	+	+
+	**0**	**0**
+	**0**	+
+	+	**0**
+	+	+

Note that if we replace **+** by **1** then these possibilities are an ordered list corresponding to the base two expansions of the integers 0 through 7. Our next step is to add two columns to this table, one of which indicates whether **x ∈ A ∩ (B ∪ C)** in the given case and the other of which gives reasons for this conclusion.

x ∈ A	**x ∈ B**	**x ∈ C**	**x ∈ A ∩ (B ∪ C)**	*Reason(s)*
0	**0**	**0**	**0**	**x ∉ A**
0	**0**	+	**0**	**x ∉ A**
0	+	**0**	**0**	**x ∉ A**
0	+	+	**0**	**x ∉ A**
+	**0**	**0**	**0**	**x ∉ B ∪ C**
+	**0**	+	+	**x ∈ A & x ∈ C**
+	+	**0**	+	**x ∈ A & x ∈ B**
+	+	+	+	**x ∈ A & x ∈ B**

We next carry out the same process for $x \in (A \cap B) \cup (A \cap C)$:

$x \in A$	$x \in B$	$x \in C$	$x \in (A \cap B) \cup (A \cap C)$	*Reason(s)*
0	0	0	0	$x \notin A$
0	0	+	0	$x \notin A$
0	+	0	0	$x \notin A$
0	+	+	0	$x \notin A$
+	0	0	0	$x \notin B \cup C$
+	0	+	+	$x \in A$ & $x \in C$
+	+	0	+	$x \in A$ & $x \in B$
+	+	+	+	$x \in A$ & $x \in B$

In both instances we see that **x** belongs to the set under consideration if and only if one of the last three possibilities is true. Therefore the two sets, namely $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$, must be equal. This proves the first distributive law.

Of course, it is possible to approach the second distributive law similarly. We shall not carry out the details here (the latter is left to the reader as an exercise), but we note that $x \in S$ belongs to the sets under consideration in this situation if and only if one of the last five possibilities in the first table is true.■

This completes the discussion of Theorem 1, so we shall proceed to the identities in Theorem 2 involvling complementation.

Double negative law. Let **P** be the statement that $x \in A$, and let **Q** be the statement that $x \in S$. Since **A** is a subset of **S** we know that **P** is equivalent to $P \wedge Q$. The statement $x \in A'$ is then given by $Q \wedge (\neg P)$, and the statement $x \in (A')'$ is then given by $Q \wedge \neg [Q \wedge (\neg P)]$. We then have the chain of logical equivalences

$$Q \wedge \neg [Q \wedge (\neg P)] \Leftrightarrow Q \wedge [(\neg Q) \vee (\neg \neg P)]$$

$$Q \wedge [(\neg Q) \vee (\neg \neg P)] \Leftrightarrow Q \wedge [(\neg Q) \vee P]$$

$$Q \wedge [(\neg Q) \vee P] \Leftrightarrow [Q \wedge (\neg Q)] \vee [Q \wedge P]$$

$$[Q \wedge (\neg Q)] \vee [Q \wedge P] \Leftrightarrow Q \wedge P \Leftrightarrow P$$

which show that $x \in (A')' \Leftrightarrow x \in A$.

Here is a nonsymbolic approach: In this case the two possibilities are given by the statements **$x \in A$** and **$x \notin A$.** In the first case we know that **$x \in A$** implies **$x \notin A'$,** which in turn implies that **$x \notin (A')'$** is false or equivalently that **$x \in (A')'$** is true**.** To prove the converse direction, note that **$x \in (A')'$** implies **$x \notin A'$,** which we know is equivalent to **$x \in A$.** This completes the argument in the first case. In the second case, we know that **$x \notin A$** implies **$x \in A'$,** which in turn implies **$x \notin (A')'$.** Conversely, if the latter is true then **$x \in A'$,** which in turn is equivalent to **$x \notin A$.** Thus in all cases we see that **$x \in (A')' \Leftrightarrow x \in A$.**

Complementation laws. If either **$x \in A$** or **$x \in A'$** then we also have **$x \in S$,** so that **$A \cup A'$** is contained in **S.** Conversely, if **$x \in S$,** then we either have **$x \in A$** or else we have **$x \notin A$,** or equivalently **$x \in A'$,** so that **$x \in A \cup A'$.** Therefore **$A \cup A' = S$.** Next, if both **$x \in A$** and **$x \in A'$** then we have **$x \in A$** and **$x \notin A$,** which is impossible. Therefore there cannot be any **x** in the intersection and hence it must be empty.

De Morgan's laws. Let **P** and **Q** be the statements **$x \in A$** and **$x \in B$,** and let **R** be the statement **$x \in S$.** The statement **$x \in (A \cup B)'$** is then given by **$R \wedge [\neg (P \vee Q)]$,** and we can then chase the string of equivalences

$$R \wedge [\neg (P \vee Q)] \quad \Leftrightarrow \quad R \wedge (\neg P \wedge \neg Q)$$

$$R \wedge (\neg P \wedge \neg Q) \quad \Leftrightarrow \quad (R \wedge \neg P) \wedge (R \wedge \neg Q)$$

to see that **$x \in (A \cup B)' \Leftrightarrow x \in A' \cap B'$.** Likewise, the statement **$x \in (A \cap B)'$** is given by **$R \wedge [\neg (P \wedge Q)]$,** and we can then chase the string of equivalences

$$R \wedge [\neg (P \wedge Q)] \quad \Leftrightarrow \quad R \wedge (\neg P \vee \neg Q)$$

$$R \wedge (\neg P \vee \neg Q) \quad \Leftrightarrow \quad (R \wedge \neg P) \vee (R \wedge \neg Q)$$

to see that **$x \in (A \cap B)' \Leftrightarrow x \in A' \cup B'$.**

Once again we shall give another proof of this without using propositional equivalences by breaking things down into cases. Here there are four possibilities, depending on whether each of the basic statements **$x \in A$** and **$x \in B$** is true. To save space we shall proceed directly to determine whether or not **$x \in (A \cap B)'$** in the respective cases.

$x \in A$	$x \in B$	$x \in (A \cap B)'$	*Reason(s)*
0	0	+	$x \notin A$ & $x \notin B$
0	+	+	$x \notin A$
+	0	+	$x \notin B$
+	+	0	$x \in A$ & $x \in B$

If one carries out the analogous procedure with **x** $\in$ **A′** $\cup$ **B′** replacing the third column, exactly the same result is obtained (with the same reasons in each case). Therefore, each of the separate statements **x** $\in$ **(A** $\cap$ **B)′** and **x** $\in$ **A′** $\cup$ **B′** holds in the first three of the four possibilities, and accordingly the two sets under consideration must be equal. This proves the second of De Morgan's Laws.

A similar approach yields the first of De Morgan's Laws; in this situation **x** $\in$ **S** belongs to the sets under consideration, which are **(A** $\cup$ **B)′** and **A′** $\cap$ **B′,** in only the first of the four possibilities.■

III.2: Ordered pairs and products

(Halmos, §§ 3, 6; Lipschutz, §§ 3.1 – 3.2)

We shall introduce ordered pairs axiomatically, following an approach outlined on page 25 of Halmos (see the paragraph beginning near the middle of the page). As shown the preceding discussion on pages 23 – 24 of Halmos, it is possible to derive our axiom(s) as consequences of the other assumptions introduced up to this point. There will be further discussion of efficient and irredundant systems of axioms later in these notes.

EXISTENCE OF ORDERED PAIRS. *Given two set-theoretic objects* **a** *and* **b,** *there is a set-theoretic construction which yields an*

ordered pair **(a, b)**

which has the fundamental property

(a, b) = (c, d) *if and only if* **a = c** *and* **b = d.**

Given two classes **A** and **B,** the ***Cartesian product*** **A** $\times$ **B** is defined to be the collection of all ordered pairs **(a, b)** where **a** $\in$ **A** and **b** $\in$ **B.** This collection is also called the ***direct product*** of the two sets **A** and **B.**

Nonmathematical example. If set **V** is the set of playing card values **{ A, K, Q, J,** 10**,** 9**,** 8**,** 7**,** 6**,** 5**,** 4**,** 3**,** 2 **}** and set **S** is the set of playing card suits **{ ♠, ♥, ♦, ♣ },** then the Cartesian product **V** $\times$ **S** corresponds to the standard deck of 52 playing cards:

{ (A, ♠), (K, ♠), ..., (2, ♠), (A, ♥), ..., (3, ♣), (2, ♣) }

Historical remarks. Clearly the name ***Cartesian product*** is an allusion to the well known work of R. Descartes (1596 – 1650) on introducing algebraic coordinates into geometry. This usage is somewhat ironic because Descartes himself did not explicitly use ordered pairs of numbers to represent points in his writings on coordinate geometry.

The latter are formally just part of one addendum, *La Géométrie*, to his major work, *Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences* (Discourse on the Method of Correctly Reasoning and Seeking Truth in the Sciences). However, the name ***Cartesian product*** has stuck and is now unlikely to be changed. A detailed discussion about exactly how Descartes and several others, including P. de Fermat (1601 – 1665), introduced coordinates into geometry during the 17^{th} century, and the significance of various individuals' contributions, is far beyond the scope of these notes, but some information on these topics is given on pages 370 and 375 – 376 of Burton, and the following references on the history of mathematics provide still more details:

> C. B. Boyer, ***A History of Mathematics.*** (Revised reprinting of the second edition, with a foreword by Isaac Asimov. Revised and with a preface by U. C. Merzbach.) *John Wiley & Sons, Inc., New York,* 1991. ISBN: 0–471–54397–7. [See in particular pages 345 – 346.]
>
> C. B. Boyer, ***History of Analytic Geometry.*** *Dover Publications, New York,* 2004. ISBN: 0–486–43832–5.
>
> M. Kline, ***Mathematical Thought from Ancient to Modern Times.*** Oxford University Press, Oxford, UK, 1972. ISBN: 0–195–01496–0.

In order to work effectively with Cartesian products like **A** × **B** we need the following axiom.

CARTESIAN PRODUCT PROPERTY. *If* **A** *and* **B** *are sets*, *then so is* **A** × **B.**

If one uses the construction for ordered pairs on pages 23 – 24 of Halmos, then this axiom follows immediately (see the discussion on page 24).

In our setting (and the development in Halmos) it follows immediately that if **C** and **D** are subsets of **A** and **B** respectively, then **C** × **D** is a subset of **A** × **B** (compare the final sentence on page 25 of Halmos).

It is important to recognize that the products **B** × **A** and **A** × **B** are not necessarily equal. In fact, we have the following result:

Proposition 1. *If* **A** *and* **B** *are nonempty sets, then we have* **B** × **A** = **A** × **B** *if and only if* **A** = **B.**

Proof. If **A** = **B** then we trivially have **A** × **B** = **A** × **A** = **B** × **A.** Conversely, suppose we have **B** × **A** = **A** × **B.** Let **b** ∈ **B.** Then for each **x** ∈ **A** we have **(b, x)** ∈ **B** × **A** = **A** × **B,** which means that **b** ∈ **A.** Thus we have shown that **B** is contained in **A.** Similarly, let **a** ∈ **A.** Then for each **y** ∈ **B** we have **(y, a)** ∈ **B** × **A** = **A** × **B,** which means that **a** ∈ **B.** Thus we have shown that **A** is contained in **B.** Combining these, we conclude that **A** = **B.**■

Here is another elementary result on Cartesian products. The proof is left to the reader as an exercise.

<u>Proposition 2.</u> *If* **a** *and* **b** *are sets, then* $\{ \mathbf{a} \} \times \{ \mathbf{b} \} = \{ (\mathbf{a}, \mathbf{b}) \}$.■

A few simple formal identities involving Cartesian products, unions, intersections and complements are listed at the bottom of page 25 in Halmos, and more are given in the exercises for this section.

<u>Notational remark.</u> As noted on page 13 of the book by Munkres, the notation **(a, b)** for an ordered pair has an entirely different meaning than the use of **(a, b)** to denote an open interval in the real numbers; *i.e.*, all real numbers **x** such that $\mathbf{a} < \mathbf{x} < \mathbf{b}$. Usually it is very clear from the context which meaning should be given to **(a, b)** but there are some exceptions. The terminology $\mathbf{a} \times \mathbf{b}$ is sometimes used (for example, in Munkres), but this can also lead to conflicts of various sorts so we shall avoid it.

III.3 : Larger constructions

(Halmos, §§ 3, 5 – 6, 9; Lipschutz, §§ 1.9, 3.1 – 3.2, 5.1 – 5.2)

Usually the sets constructed in the preceding two sections are not much larger than the objects from which they are constructed. In this section we shall discuss some basic constructions which generally yield much larger examples

Power sets

In Section **II.1** we noted that sets may themselves be elements of other sets. The following quoted passage from page 11 of Munkres, ***Topology*** (full citation below), explains the main idea in nonmathematical terms.

> *The objects belonging to a set may be of any sort. One can consider … the set of all decks of playing cards in the world … [which] illustrates a point we have not yet mentioned; namely, the objects belonging to a set may* ***themselves*** *be sets. For a deck of cards is itself a set, one consisting of pieces … [and] the set of all decks of cares in the world is thus a set whose elements themselves are sets.*
>
> [***<u>Source:</u>*** J. R. Munkres, ***Topology*** (Second Edition). Prentice – Hall, Upper Saddle River NJ, 2000. ISBN: 0 – 13 – 18129 – 2.]

We may state this principle more formally as follows:

<u>POWER SET PROPERTY.</u> *If* **S** *is a set, then the collection* **P(S)** *of all subsets of* **S** *is also a set.*

This set of all subsets is often called the ***power set*** for reasons that will be explained in the next unit.

Note that union, intersection and complementation define algebraic operations on **P(S)** which satisfy the identities described above. In some ways union and addition behave like addition and multiplication, but a check of the Boolean algebra identities also shows some important differences (for example, the idempotent laws and the fact that one has an extra distributive law).

<u>Examples.</u> If **S** is the set **{1, 2, 3 }** then there are precisely $\mathbf{8 = 2^3}$ subsets in **P(S),** and they are all listed below:

Ø, {1}, {2}, {3}, {1, 2 }, {1, 3 }, {2, 3 }, {1, 2, 3 }

If **T** is the set **{1, 2, 3, 4 }** then there are precisely $\mathbf{16 = 2^4}$ subsets in **P(S),** and they may be obtained from the list above by **(*i*)** taking the eight sets in this list, **(*i i*)** adding the element **4** to each of the eight sets in this list. Clearly one could continue in this fashion to list the subsets of **{1, 2, 3, 4, 5 }** and even larger finite sets; in particularm the set of all subsets of **{1, … , n }** contains $\mathbf{2^n}$ elements**.**

Note that the power set construction can be iterated, yielding sets such as **P(P(S)), P(P(P(S))),** and so forth.

<u>Example.</u> If **S** is the set **{1}** then **P(P(S))** consists of the objects **Ø, { Ø } , { {1} } ,** and **P(S).**

Larger unions and intersections

We have already noted that the importance of set theory is directly tied to its usefulness in studying infinite collections of objects. In particular, it is often necessary to consider unions and intersections of more than two sets at a time. Therefore we shall need an axiom to guarantee that reasonable infinite unions and intersections will determine sets.

<u>AXIOM OF UNIONS.</u> *If* **A** *is a set and* $**(A)** *is the collection of all* **x** *such that* **x ∈ B** *for some* **B ∈ A,** *then* **$(A)** *is also a set***.**

<u>Nonmathematical example.</u> If **A** represents the set of all decks of playing cards as above, then **$(A)** is just the set of all cards belonging to these decks.

Normally one writes **$(A)** in another notation that is more suggestive of taking unions; for example, we frequently use expressions like $\cup$ **{B | B ∈ A }** or $\cup_{\mathbf{B \in A}}$ **B.** This set is often called the ***union*** of all the sets **B** in the collection **A.** Our choice of the symbol **$** is motivated by typographical limitations in the word processing program used to create these notes.

There is also a corresponding notion of intersection.

Proposition 1. *If* **A** *is a* ***nonempty*** *set then there is a set*

$$\{ x \in \$(A) \mid x \in B \text{ for } \underline{\textbf{all}}\ B \in A \}$$

which is called the ***intersection*** *of all the sets* **B** *in the collection* **A** *and written in the forms* $\cap \{B \mid B \in A\}$ *or* $\cap_{B \in A} B$.

This result is an immediate consequence of the Axiom of Specification. The reasons for assuming **A** is nonempty are discussed on pages 18 – 19 of Halmos; for most purposes it is simply enough to understand that there are some annoying (but not serious) logical complications if we allow the possibility $A = \varnothing$.■

Further topics involving large unions and intersections will be covered in Section **VII.1.**

Products of more than two sets

We have already described the product of two sets in terms of ordered pairs. More generally, one can also discuss ordered **n** – tuples of the form $(x_1, \dots, x_n)$ and define an **n** – fold Cartesian product $A_1 \times \dots \times A_n$ which will be the collection of all ordered **n** – tuples $(x_1, \dots, x_n)$ such that $x_k \in A_k$ for all **k** between 1 and **n.** There will be a few references to such constructions in the next few units, and in Section **V.1** of these notes we shall show that one can even construct Cartesian products of infinite lists of sets. We shall state the explicit generalizations from ordered pairs to ordered **n** – tuples below**.** Everything is a straightforward extension of the previous discussion for **n = 2.**

EXISTENCE OF ORDERED n – TUPLES. *Let n be a positive integer. Given a sequence of* **n** *set-theoretic objects* $a_1, \dots, a_n$ *there is a set-theoretic construction which yields an*

ordered **n** *– tuple* $(a_1, \dots, a_n)$

which has the fundamental property

$$(a_1, \dots, a_n) = (b_1, \dots, b_n) \text{ if and only if } a_i = b_i \text{ for all } i.$$

Given **n** classes $A_1, \dots, A_n$ the ***Cartesian product*** $A_1 \times \dots \times A_n$ is defined to be the collection of all ordered **n** – tuples $(a_1, \dots, a_n)$ where $a_i \in A_i$ for all **i.** This collection is also called the ***direct product*** of the classes**.**

GENERALIZED CARTESIAN PRODUCT PROPERTY. *If* $A_1, \dots, A_n$ *are sets, then so is* $A_1 \times \dots \times A_n$.

Proposition 2. *If* $a_1, \dots, a_n$ *are sets, then* $\{a_1\} \times \dots \times \{a_n\} = \{(a_1, \dots, a_n)\}$.■

III.4 : A convenient assumption

(Halmos, § 2; Lipschutz, § 1.12)

In Unit **II,** the following question arose in connection with Russell's paradox**:**

Is it possible to have an object **z** *in set theory such that* $\mathbf{z} \in \mathbf{z}$**?**

We might not expect something like this to happen when we discuss collections of ordinary objects, but nothing that we have said thus far eliminates such possibilities from set theory. The purpose of this section is to note that the latter do not arise in the most widely used approaches to set theory and to explain how this is done, mainly from the naïve point of view.

There are also many questions of a similar nature that can be formulated. Here is one crucial example:

Is it possible to have objects **u** *and* **v** *such that* $\mathbf{u} \in \mathbf{v}$ *and* $\mathbf{v} \in \mathbf{u}$**?**

These and other questions were considered early in the 20^{th} century, and the key general observation was first noticed by D. Mirimanov (1861 – 1945) in 1917. The following equivalent formulation was given by J. von Neumann in the 1920s.

<u>AXIOM OF FOUNDATION.</u> *For each nonempty set* **x** *there is a set* **y** *such that* $\mathbf{y} \in \mathbf{x}$ *and* $\mathbf{y} \cap \mathbf{x} = \emptyset$**.**

This assumption, which is also known as the **<u>AXIOM OF REGULARITY</u>**, can be rephrased entirely in terms of words as follows:

*<u>Every nonempty set is disjoint from **at least one** of its elements.</u>*

The relation between this axiom and the condition in Russell's paradox is contained in the following result.

<u>Proposition 1.</u> *For every set* **z** *we have* $\mathbf{z} \notin \mathbf{z}$**.**

<u>Proof.</u> Let **x** be the set $\{ \mathbf{z} \}$**,** so that the Axiom of Foundation implies the existence of some **y** such that $\mathbf{y} \in \mathbf{x}$ and $\mathbf{y} \cap \mathbf{x} = \emptyset$**.** Since **x** contains only the element **z,** it follows that **y** must be equal to **z,** and thus the condition $\mathbf{y} \cap \mathbf{x} = \emptyset$ translates to the condition $\mathbf{z} \cap \{ \mathbf{z} \} = \emptyset$**.** The latter is in turn equivalent to $\mathbf{z} \notin \mathbf{z}$**.**■

Similarly, we can use the Axiom of Foundation to show that the answer to the second question is also ***<u>NO</u>*****.**

<u>Proposition 2.</u> *If* **z** *and* **w** *are sets***,** *then either* $\mathbf{z} \notin \mathbf{w}$ *or* $\mathbf{w} \notin \mathbf{z}$ *is true (and both might be true)***.**

Proof. It will suffice to show that if $\mathbf{z} \in \mathbf{w}$ then $\mathbf{w} \notin \mathbf{z}$; therefore we shall suppose that $\mathbf{z} \in \mathbf{w}$ is true. Let $\mathbf{x}$ be the set $\{\mathbf{z}, \mathbf{w}\}$, so that the Axiom of Foundation implies the existence of some $\mathbf{y}$ such that $\mathbf{y} \in \mathbf{x}$ and $\mathbf{y} \cap \mathbf{x} = \emptyset$. It follows immediately that either $\mathbf{y} = \mathbf{z}$ or $\mathbf{y} = \mathbf{w}$. If $\mathbf{y} = \mathbf{z}$, then $\mathbf{z} \in \mathbf{w}$ would imply that $\mathbf{y}$ and $\mathbf{x}$ have $\mathbf{z}$ in common, which contradicts the fundamental condition on $\mathbf{y}$, so we must have $\mathbf{z} \notin \mathbf{w}$. Likewise, if $\mathbf{y} = \mathbf{w}$, then $\mathbf{w} \in \mathbf{z}$ would imply that $\mathbf{y}$ and $\mathbf{x}$ have $\mathbf{w}$ in common, which contradicts the fundamental condition on $\mathbf{y}$, so we must have $\mathbf{w} \notin \mathbf{z}$. Therefore in all cases at least one of the statements $\mathbf{z} \notin \mathbf{w}$ or $\mathbf{w} \notin \mathbf{z}$ must be true.■

More general consequences along these lines are discussed and proved on pages 95 – 96 of the book by Goldrei (see the beginning of Unit **I** for full bibliographic information). We shall merely state Mirimanov's original formulation of the property and one generalization (both without proofs; see pages 95 – 96 of Goldrei for details):

Mirimanov's Axiom of Foundation. *There are no sequences of sets* $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots$ *such that* $\mathbf{A}_k \in \mathbf{A}_{k+1}$ *for all* $\mathbf{k}$.

Special case. *There are no sequences of sets* $\mathbf{A}_1, \mathbf{A}_2, \ldots \mathbf{A}_n$ *such that* $\mathbf{A}_k \in \mathbf{A}_{k+1}$ *for all* $\mathbf{k}$ *and* $\mathbf{A}_n \in \mathbf{A}_1$ *(i.e., there are* ***no finite length*** $\in$ – ***cycles****).*

The second statement follows from the first by reductio ad absurdum, for if a finite sequence of the described type existed, then one could extend it to an infinite sequence as follows: Given an arbitrary positive integer $\mathbf{m}$, use long divising to write $\mathbf{m} = \mathbf{q}\,\mathbf{n} + \mathbf{r}$ where $\mathbf{0} \leq \mathbf{r} < \mathbf{n}$, and set $\mathbf{A}_m = \mathbf{A}_r$. By construction this is a ***periodic*** or ***repeating*** sequence such that $\mathbf{A}_m = \mathbf{A}_{m+n}$ for all $\mathbf{m}$.■

FOOTNOTE. Biographical information on D. Mirimanov (also spelled Mirimanoff) is available at the following online site:

http://www.numbertheory.org/obituaries/OTHERS/mirimanoff.html

(Unfortunately, the chronology for his life is in French, but the main items in it should be decipherable, and standard Internet translation software should work reasonably well for this material.)

Should one really assume the Axiom of Foundation?

A few mathematicians have varying degrees of reservations about assuming the Axiom of Foundation, but most accept it both because **(1)** as we have noted it is convenient to do so, **(2)** the introduction of this assumption does not lead to any logical contradictions by itself. The second point requires some explanation. Later in these notes we shall discuss the following important question:

Can we be certain that our logical framework for mathematics is entirely free of contradictions?

Unfortunately, the answer is ***NO*****,** and in fact the answer is no for any system that involves infinite objects like the basic number systems such as the positive integers or the real numbers. However, if there is a logical contradiction in the standard framework for mathematics which includes the Axiom of Foundation, then fundamental results of K. Gödel (1906 – 1978) imply that there is already a logical contradiction in the framework if one drops this assumption. Further information on this and related topics will appear in Unit **VII** when we introduce the Axiom of Choice**.**

Here are some online references for approaches to set theory that do not assume the Axiom of Foundation**:**

http://en.wikipedia.org/wiki/Non-well-founded_set_theory

http://en.wikipedia.org/wiki/Axiomatic_set_theory

A more extensive (and quite advanced) reference for set theory without the Axiom of Foundation is *Non – well – founded sets*, by P. Aczel, which is available at the following online site**:**

http://standish.stanford.edu/pdf/00000056.pdf

Historical remarks

With the emergence of Russell's paradox, most mathematicians and logicians from that time concluded that set theory probably should not contain objects for which $\mathbf{x} \in \mathbf{x}$ or pairs of objects such that $\mathbf{x} \in \mathbf{y}$ and $\mathbf{y} \in \mathbf{x}$. Russell's approach to eliminating such phenomena was to introduce a ***theory of types*****,** in which sets have well – defined ***types*** or ***levels*** such that the level a set should exceed the level of its elements. Such a theory will not contain objects with the undesirable properties described above, and it also will not allow the other sorts of paradoxes that arose near the beginning of the 20th century. The theory of types played a central role in Russell's work with A. N. Whitehead (1861 – 1947) to create a logically unassailable foundation for mathematics, which culminated in their massive and ambitiously titled *Principia Mathematica*, a work consisting of nearly 2000 pages which was published during the period 1910 – 1913 and whose title echoes Isaac Newton's monumental *Philosophiæ Naturalis Principia Mathematica*. The amount of detail in the work is illustrated by one frequently stated piece of trivia; namely, a proof that **"$1 + 1 = 2$"** does not appear until a few hundred pages into the book. The relevant page is depicted at following online site:

http://www.idt.mdh.se/~icc/1+1=2.htm

Some online references for the Russell – Whitehead *Principia* and a bibliographic reference are listed below. These include biographical references for the coauthors written from the perspective of philosophy as well as a biography of G. Frege (1848 – 1925), whose writings and ideas exerted a strong influence on the work of Russell and Whitehead.

B. Russell and A. N. Whitehead, ***Principia Mathematica***
(2nd Rev. Ed.), Cambridge University Press, Cambridge,
UK, and New York, 1962. ISBN: 0–521–06791–X.

http://en.wikipedia.org/wiki/Alfred_North_Whitehead

http://plato.stanford.edu/entries/whitehead/

http://plato.stanford.edu/entries/russell/

http://plato.stanford.edu/entries/frege/

http://plato.stanford.edu/entries/principia-mathematica/

One disadvantage of the theory of types is the amount of duplication it requires; at each level one has an exact copy of the previous level. In some sense, von Neumann's Axiom of Foundation and the introduction of classes **"**too big**"** to be sets is a drastic simplification of the system of levels in the theory of types which still eliminates highly uncomfortable possibilities like $\mathbf{x \in x}$**.**