Addendum to III.3 — Unions and intersections over subfamilies

This is a written version of some in-class remarks that are not in the notes.

The following result describes what happens to unions and intersections of families of sets if one passes from a family $\mathbf{F}$ to a subfamily $\mathbf{G}$ such that $\mathbf{G} \subset \mathbf{F}$.

Theorem III.3.1A. *Let $\mathbf{F}$ be a family of sets, and let $\mathbf{G}$ be a subfamily of $\mathbf{F}$. Then we have*

$$\bigcup \{ B \mid B \in \mathbf{G} \} \quad \subset \quad \bigcup \{ B \mid B \in \mathbf{F} \} \ .$$

Furthermore, if $\mathbf{F}$ and $\mathbf{G}$ are nonempty then we also have

$$\bigcap \{ B \mid B \in \mathbf{F} \} \quad \subset \quad \bigcap \{ B \mid B \in \mathbf{G} \} \ .$$

PROOF. Suppose that $x \in B_0$ for some $B_0 \in \mathbf{G}$. Then we also know that $B_0 \in \mathbf{F}$, and therefore x must also belong to $\cup \{ B \mid B \in \mathbf{F} \}$. Suppose now that $\mathbf{F}$ and $\mathbf{G}$ are nonempty and that $x \in \cap \{ B \mid B \in \mathbf{F} \}$. If $C \in \mathbf{G}$, then $C \in \mathbf{F}$, and therefore if $x \in B$ for every $B \in \mathbf{F}$ then certainly $x \in B$ for every $B \in \mathbf{G}$. But this means that $\cap \{ B \mid B \in \mathbf{F} \}$ is contained in $\cap \{ B \mid B \in \mathbf{G} \}$.■