A rapid review of cardinal numbers

This material is normally covered near the end of Mathematics 144, and we are summarizing the facts that are needed to work exercises in this course. Further details appear in Sections VI.3, VI.4 and VII.1 of the course directory file `set-theory-notes.pdf`.

Cardinal numbers

If S is a set, then there is an object $|S|$, called the *cardinal number* or *cardinality* of S, such that $|S| = |T|$ if and only if there is a 1–1 correspondence $f : S \to T$. The cardinal numbers of sets have a partial ordering with $|S| \leq |T|$ if and only if there is a 1–1 (but not necessarily onto) function $f : S \to T$. It follows immediately that this relation is reflexive (take the identity map from S to itself) and transitive (given 1–1 maps $f : S \to T$ and $g : T \to U$, the composite map $g \circ f$ is also 1–1); on the other hand, the antisymmetry property

$$|S| \quad \leq \quad |T| \quad \text{and} \quad |T| \quad \leq \quad |S| \quad \text{implies} \quad |S| \quad = \quad |T|$$

is a nontrivial result known as the **Schröder-Bernstein Theorem** (see page 131 of `set-theory-notes.pdf` for a proof). A standard assumption in set theory (the *Axiom of Choice*) implies that this partial ordering is in fact a linear ordering (see Theorem VII.1.4 on page 160 of `set-theory-notes.pdf`), but in this course we do not need to know whether or not the ordering is linear.

For a nonempty finite set A, the cardinal number $|A|$ is the unique positive integer n such that there is a 1–1 correspondence from A to $\{1, \cdots, n\}$; by convention, the cardinality of the empty set $\emptyset$ is equal to 0. The crucial fact about **transfinite cardinal numbers** is that different infinite sets may have different cardinal numbers.

Formal properties of cardinal numbers

We have the following rules for working with cardinal numbers, and they will suffice for solving the exercises in this course (this list is not exhaustive; additional rules are stated in `set-theory-notes.pdf`). Following standard notational conventions, we shall denote the cardinality of the set $\mathbb{N}$ of nonnegative integers by $\aleph_0$, which is verbalized as "aleph-null" (*Footnote:* The symbol aleph, written $\aleph$, is the first letter of the Hebrew alphabet).

1. If S is an infinite set, then $|S| \geq \aleph_0$.

2. If $f : S \to T$ is onto, then $|S| \geq |T|$.

3. If $|A| = |B|$ and $|C| = |D|$, then $|A \times B| = |C \times D|$.

4. If k is a positive integer and A^k denotes a product of k copies of A, then $\aleph_0 = |\mathbb{N}^k| = |\mathbb{Z}^k| = |\mathbb{Q}^k|$.

5. If A is a set and $\mathcal{P}(A)$ denotes the set of all subsets of A, then $|\mathcal{P}(A)| > A$ (*i.e.*, we have $|\mathcal{P}(A)| \geq A$ but not $|\mathcal{P}(A)| = A$).

6. If k is a positive integer, then $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}^k| = |\mathbb{C}^k|$. In particular, we have $|\mathbb{R}| > |\mathbb{Q}|$.

7. If $|A| = |B|$ then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

Here are the references for proofs of these facts in `set-theory-notes.pdf`:

Rule 1.	Proposition VI.3.2 on page 131 and Theorem VII.1.1 on page 159.
Rule 2.	Theorem VII.1.5 on page 161.
Rule 3.	Proposition VI.4.2 on page 134.
Rule 4.	Corollary VI.4.9 on pages 137–138.
Rule 5.	Theorem VI.4.4 on page 134.
Rule 6.	Theorems VI.4.12 and VI.4.13 on pages 139–140.
Rule 7.	Proposition VI.4.2 and VI.4.3 on page 134.

The preceding rules have the following useful consequence:

PROPOSITION 1. *Let J be a nonempty subset of the nonnegative integers, and suppose that $\{A_j \mid j \in J\}$ is a family of pairwise disjoint sets such that $|A_j| \leq |\mathbb{R}|$ for all $j \in J$. If $A = \cup_j A_j$, then $|A| \leq |\mathbb{R}|$.*

Proof. For each j let $f_j : A_j \to \mathbb{R}$ be a 1–1 mapping. Define $F : A \to \mathbb{R}^2$ such that if $x \in A_j$ then $F(x) = (f_j(x), j)$. This map is well-defined because x can belong to only one subset A_j. It is also 1–1 because $F(x) = F(y)$ implies the second coordinates are equal, so that $x, y \in A_j$ for some j; therefore $F(x) = F(y)$ also implies $f_j(x) = f_j(y)$. Since f_j is a 1–1 mapping, it follows that $x = y$. Therefore we have $|A| \leq |\mathbb{R}^2|$, and since $|\mathbb{R}^2| = |\mathbb{R}|$ by Rule 6, it follows that $|A| \leq |\mathbb{R}|$.■

A simple but important application of tranfinite cardinals

The following is a typical but fundamental example of how the rules for manipulating transfinite cardinals are used in point set theory:

PROPOSITION 2. *If U is a nonempty subset of $\mathbb{R}^n$ for some $n \geq 1$, then $|U| = |\mathbb{R}|$.*

Proof. The inclusion $U \subset \mathbb{R}^k$ and Rule 6 imply that $|U| \leq |\mathbb{R}^n| = |\mathbb{R}|$, so by the Schröder-Bernstein Theorem it is enough to prove that $|\mathbb{R}| \leq |U|$.

If $x \in U$, then there is some $\varepsilon > 0$ such that $N_\varepsilon(x; d_\infty) \subset U$, where the left hand side denotes the ε- neighborhood with respect to the d_∞-metric on $\mathbb{R}^n$; if we write x in terms of its coordinates as $(x_1, \cdots, x_n)$, then we have

$$N_\varepsilon(x; d_\infty) \quad = \quad \prod_{i=1}^{n} (x_i - \varepsilon, x_i + \varepsilon) \ .$$

For each interval in the product there is a 1–1 correspondence with the real line such that $t \in \mathbb{R}$ corresponds to

$$u \quad = \quad x_i + \frac{\varepsilon\, t}{1 + |t|}$$

(if $v = (u - x_i)/\varepsilon$, then the inverse function is given by $t = v/(1 - |v|)$), so by Rules 3 and 6 the cardinality of this set is equal to $|\mathbb{R}|$. Therefore, since $N_\varepsilon(x; d_\infty) \subset U$ we have

$$|\mathbb{R}| \quad = \quad |N_\varepsilon(x; d_\infty)| \quad \leq \quad |U|$$

and by the previous paragraph this completes the proof.■