Mathematics 145A, Winter 2014, Examination 2

Answer Key

1. [*25 points*] Let $\mathcal{U}$ be the usual topology on the real numbers $\mathbb{R}$, and let $\mathcal{V}$ be the topology consisting of the empty set, the entire real line, and all open rays of the form $(c, +\infty)$, where $c \in \mathbb{R}$.

(i) Let $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{V})$ and $g : (\mathbb{R}, \mathcal{V}) \to (\mathbb{R}, \mathcal{U})$ be the respective identity mappings. Which (if any) of the maps f, g is/are continuous?

(ii) Show that $(\mathbb{R}, \mathcal{V})$ is not Hausdorff. [*Hint:* Given a pair of distinct points in $\mathbb{R}$, one of them is larger than the other. What are the open subsets containing a given point?]

SOLUTION

(i) The topology $\mathcal{V}$ is strictly contained in $\mathcal{U}$, for every $\mathcal{V}$-open subset is $\mathcal{U}$-open, but $\mathcal{V}$ does not contain an open interval of the form (a, b) where $b < +\infty$. Therefore the identity map $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{V})$ satisfies the condition for continuity but the inverse identity map $g : (\mathbb{R}, \mathcal{V}) \to (\mathbb{R}, \mathcal{U})$ does not.■

(ii) Since the nonempty open subsets for $\mathcal{V}$ have the form $(c, +\infty)$, if $x < y$ and $x \in W$ for some W in $\mathcal{V}$, then we also have $y \in W$. Therefore every open subset containing x contains y. If $(\mathbb{R}, \mathcal{V})$ were a Hausdorff space one could find disjoint open U and V such that $x \in U$ and $y \in V$. Since this does not happen in $(\mathbb{R}, \mathcal{V})$, the latter cannot be Hausdorff.■

Alternatively, we know that a one point subset $\{x\}$ is never closed because its complement $W = (-\infty, x) \cup (x, +\infty)$ is never open; in particular, we have $x - 1 \in W$ but $x \notin W$, so the condition in the first sentence of the preceding paragraph is not satisfied. If $(\mathbb{R}, \mathcal{V})$ were Hausdorff, then all complements of one point subsets would be open, and since this is not the case the space $(\mathbb{R}, \mathcal{V})$ is not Hausdorff.■

2. [*25 points*] (i) Show that the set of limit points for the closed interval $[0,1]$ is equal to all of $[0,1]$.

(ii) If A is a subset of a topological space X and A is not open, explain why the interior of A must be a proper subset of A.

SOLUTION

(i) First of all, since $[0,1]$ is a closed set, all of its limit points must lie inside itself. To complete the exercise, we must show that every point is a limit point. For the endpoints, we can do this using sequences $a_n = \frac{1}{n}$ for the endpoint 0 (because $a_n \neq 0$ for all n and the limit is 0) and $a_n = 1 - \frac{1}{n}$ for the endpoint 1 (because $a_n \neq 1$ for all n and the limit is 1). If $0 < t < 1$ we can use the sequences $a_n = t + (1-t)/n$, whose values are never t and whose limits are t.

Alternatively, we can use the definition of limit point directly. To see that 0 and 1 are limit points, let $\varepsilon > 0$ and choose h such that $0 < h < 1$ and $h < \varepsilon/2$. Then $h \in N_\varepsilon(0;[0,1]) - \{0\}$ and $1 - h \in N_\varepsilon(1;[0,1]) - \{1\}$. To see that every other point is a limit point, suppose that $0 < t < 1$ and $\varepsilon > 0$. Without loss of generality, we may restrict attention to values of ε such that $N_\varepsilon(t) = (t-\varepsilon, t+\varepsilon)$ is contained in the open interval $(0,1)$. Then we have $t + \frac{1}{2}\varepsilon \in (N_\varepsilon(t) - \{t\}) \cap [0,1]$.■

(ii) By definition, the interior $\mathrm{Int}\, A$ of A is an open subset of X, and it is contained in A. If A is not open, then A cannot be equal to the open set $\mathrm{Int}\, A$, and therefore the inclusion of $\mathrm{Int}\, A$ in A must be proper.■

3. [*25 points*] (*i*) If X and Y are topological spaces in which one point subsets are closed, prove that $X \times Y$ (with the product topology) also has this property. [*Hint:* If E and F are closed in X and Y, why is $E \times F$ closed in the product?]

(*ii*) Let X be a topological space, and let $\mathcal{A}$ be a family of open subsets such that for each open set U and each $x \in U$ there is some V in $\mathcal{A}$ such that $x \in V$ and $V \subset U$. Prove that $\mathcal{A}$ is a base for the topology on X.

SOLUTION

(*i*) If $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the coordinate projections, they are continuous, and hence E closed in X and F closed in Y imply that

$$E \times F \quad = \quad \pi_X^{-1}[E] \ \cap \ \pi_Y^{-1}[F]$$

is closed in $X \times Y$. If $(x, y) \in X \times Y$, then the conditions in the problem imply that $\{x\}$ is closed in X and $\{y\}$ is closed in Y, and if we specialize the preceding discussion to these cases we find that

$$\{(x, y)\} \quad = \quad \{x\} \times \{y\}$$

is closed in the product.■

(*ii*) We need to show that every open set U in X is a union of open subsets from $\mathcal{A}$. Given $x \in U$, pick A_x in $\mathcal{A}$ such that $x \in A_x \subset U$. Then we have

$$U \quad = \quad \bigcup_{x \in U} \{x\} \quad \subset \quad \bigcup_{x \in U} A_x \quad \subset \quad U$$

which implies that U is the union of the open subsets A_x.■

4. [*25 points*] Let X be a topological space, and let A_1, A_2, A_3, A_4 be connected subsets such that for $A_k \cap A_{k-1}$ is nonempty for $k = 2, 3, 4$. Prove that $A_1 \cup A_2 \cup A_3 \cup A_4$ is connected.

SOLUTION

Since A_2 has a nonempty intersection with A_1 and A_3, we know that $B = A_1 \cup A_2 \cup A_3$ is connected by a result in Sutherland. But A_4 is also connected and $A_4 \cap B \supset A_4 \cap A_3 \neq \emptyset$, so another application of the same result implies that $A_1 \cup A_2 \cup A_3 \cup A_4$ is also connected.■

Note. A similar result holds for a finite sequence of connected subsets $A_1, \cdots, A_n$ such that for $A_k \cap A_{k-1}$ is nonempty for $2 \leq k \leq n$, in which case the conclusion is that the union $\cup_k A_k$ is connected. — If $n = 1$ the result is trivial, so assume it is true when there are $n-1$ sets. Then the induction hypothesis implies that $B_{n-1} = A_1 \cup \cdots \cup A_{n-1}$ is connected, and the objective is to prove that $B_n = B_{n-1} \cup A_n$ is connected.

As in the solution for the special case, we know that A_n is connected and $A_n \cap B_{n-1} \supset A_n \cap A_{n-1} \neq \emptyset$, so another application of the result cited in the first paragraph will imply that $B_n = A_n \cup B_{n-1}$ is also connected.■.