Mathematics 145A, Winter 2014, Examination 3

Answer Key

1. [*25 points*] (i) Let P be the set of all positive real numbers. What is the greatest lower bound of P?

(ii) Suppose that x, y, z, w lie in a metric space X and satisfy $d(x,y),\ d(y,z),\ d(z,w) \leq 1$. Show that $d(x,w) \leq 3$. **Extra credit, 10 points.** Give an example of a metric space X and points x, y, z, w for which $d(x,w) = 3$.

SOLUTION

(i) The greatest lower bound is zero.

Zero is a lower bound because $x \in P$ implies $x > 0$.

If $h > 0$, then h is not a lower bound because $\frac{1}{2}h < h$ and $\frac{1}{2}h \in P$.■

(ii) By two applications of the Triangle Inequality we have

$$d(x,w) \quad \leq \quad d(x,y) \ + \ d(y,w) \quad \leq \quad d(x,y) \ + \ d(y,z) \ + \ d(z,w)$$

and the inequality follows because the assumptions imply that the right hand side is less than or equal to $1+1+1 = 3$.■

Extra credit. Probably the simplest example is the real line with $x = 0$, $y = 1$, $z = 2$ and $w = 3$.■

2. [*30 points*] (i) Given a metric space X and a subset $A \subset X$, define the set $L(A)$ of limit points for A in X.

(ii) What are the boundary points of the subset of the coordinate plane given by $[a,b] \times [c,d]$? If the answer is correct, no reasons are needed, but if the answer is incorrect and suitable reasons are given, then partial credit may be given.

(iii) State the Hausdorff Separation Property.

SOLUTION

(i) The set $L(A)$ is the set of all $x \in X$ such that for each open set U containing x we have $(U - \{x\}) \cap A \neq \emptyset$.■

(ii) The set of boundary points is the union of $\{a,b\} \times [c,d]$ (the vertical edges) and $[a,b] \times \{c,d\}$ (the horizontal edges).■

(iii) Given two points u and v in the space, there are disjoint open subsets U and V such that $u \in U$ and $v \in V$.■

3. [*30 points*] (i) Let H be the set of points in the coordinate plane defined by the equation $y^2 = 1 + x^2$ (a hyperbola), and let H_+ and H_- be the sets of points where y is positive and negative respectively. Prove that H_+ and H_- are disjoint open subsets of H and $H = H_+ \cup H_-$. [*Hint:* Why are there no points in H with $y = 0$?]

(ii) Let $f : [0,1] \to Y$ be a continuous function, and let $G(f) \subset [0,1] \times Y$ be the graph consisting of all points of the form $(x, f(x))$. Prove that $G(f)$ is a connected set. [*Hint:* Show that $G(f)$ is the image of a continuous function from $[0,1]$ to $[0,1] \times Y$.]

SOLUTION

(i) H_+ is the intersection of H with the open set $\mathbb{R} \times (0, \infty)$, and H_- is the intersection of H with the open set $\mathbb{R} \times (-infty, 0)$, so these subsets are open. They are disjoint because y cannot be both positive and negative at the same time. Their union is all of H provided the latter has no points with $y = 0$; but the latter is true because $y^2 = 1 + x^2$ implies $|y| \geq 1$.

Geometrically, the two sets in the problem are the two branches of a hyperbola.∎

(ii) If F is the function from $[0,1]$ to $[0,1] \times Y$ defined by $F(t) = (t, f(t))$, then F is continuous because both of its coordinate functions are continuous. Since $[0,1]$ is connected, this means the image of F is connected. Finally, the image of F is equal to the graph of f, and if we combine this with the previous conclusions we see that the graph must be connected.∎

4. [*25 points*] If (X, d^X) and (Y, d^Y) are metric spaces, a mapping $f : X \to Y$ is said to satisfy a Lipschitz condition if there is some constant $K > 0$ such that $d^Y\big(f(u), f(v)\big) \le K \cdot d^X(u, v)$ for all $u, v \in X$. Prove that a map which satisfies a Lipschitz condition is continuous. [*Hint:* Give $x \in X$ and $\varepsilon > 0$, then for all x one can find a simple choice for δ which only involves ε and K.]

SOLUTION

Given $x \in X$ and $\varepsilon > 0$, take $\delta = \varepsilon / K$. Then $d^X(t, x) < \delta$ implies that

$$d^Y\big(f(t), f(x)\big) \quad \le \quad K \cdot d^X(t, x) \quad < \quad K \cdot \delta \quad = \quad K \cdot \frac{\varepsilon}{K} \quad = \varepsilon$$

which proves continuity at an arbitrary $x \in X$.■

5. [*20 points*] (i) Given a positive real number r, give an example of a compact subset of the real line whose diameter is greater than or equal to r.

(ii) Let A be the set of all points t on the real line such that t is rational and $0 \leq t \leq 1$. State whether or not A is compact, and give reasons for your answer.

SOLUTION

(i) There are many answers, but probably the simplest is the closed interval defined by $|x| \leq \frac{1}{2}\,r$.■

(ii) This set is not compact because it is not closed, and compact subsets of metric spaces are always closed. In particular, every irrational number in the interval $[0,1]$ is a limit point of A but does not belong to A.■

6. [*20 points*] Using the concept of compactness, explain why the polynomial function

$$p(x, y) \;=\; x^5 \;-\; 2x^4y \;+\; 3x^3y^3 \;-\; 4x^2y^3 \;+\; 5x^4y \;-\; 6y^5$$

takes a maximum value on the solid square $[0, 1] \times [0, 1]$. It is not necessary to find the point at which the function takes its maximum.

SOLUTION

The closed interval $[0, 1]$ is compact, and hence the product of this interval with itself is compact.

Since a polynomial function on the coordinate plane is continuous, the function p is continuous.

Since p is continuous, the image of the solid square, which we know is compact, must also be compact.

Since compact subsets of the real line are closed and bounded, this means that the image has a least upper bound which belongs to this image. This least upper bound is the maximum value.■