# ADDITIONAL EXERCISES FOR

# MATHEMATICS 145A — Part 1

**Fall 2014**

## 2. Notations and terminology

**0.** Given a set $X$ and a binary relation $\mathcal{R}$ on $X$, define a new binary relation $\mathcal{R}^{\#}$ on $X$ such that $x \ \mathcal{R}^{\#} \ y$ if and only if $x = y$ or there is a finite sequence $v_0, \cdots, v_m$ such that $v_0 = x$, $v_m = y$ and for each $i$ we have either $v_i \ \mathcal{R} \ v_{i+1}$ or $v_{i+1} \ \mathcal{R} \ v_i$. Prove that $\mathcal{R}^{\#}$ is an equivalence relation on $X$, and if $\mathcal{S}$ is an equivalence relation such that $x \ \mathcal{S} \ y$ whenever $x \ \mathcal{R} \ y$, then we also have $x \ \mathcal{S} \ y$ whenever $x \ \mathcal{R}^{\#} \ y$. — The latter implies that $\mathcal{R}^{\#}$ is the minimal equivalence relation on $X$ such that $x$ and $y$ are equivalent whenever $x \ \mathcal{R} \ y$, and it is called *the equivalence relation generated by* $\mathcal{R}$.

**1.** The game of chess is played on an $8 \times 8$ board with squares alternately colored black and white (or some other pair of contrasting colors). A chess player is likely to notice very quickly that a bishop can move to any square of the same color it currently occupies but cannot more to a square of the opposite color. The goal of the exercise is to give a mathematical proof of this assertion.

Here is the formal setting: Model the chessboard mathematically by the set

$$B \quad = \quad \{1,2,3,4,5,6,7,8\} \times \{1,2,3,4,5,6,7,8\}$$

so that the squares correspond to ordered pairs of points $(i, j)$ and the color of a square depends upon whether $i + j$ is even or odd. Define a binary relation $\mathcal{R}$ on $B$ such that $(i, j) \ \mathcal{R} \ (p, q)$ if $p = i + \alpha$ and $q = j + \beta$ where $\alpha, \beta \in \{-1, 1\}$ and $(p, q) \in B$ (these correspond to a bishop moving one square in any permissible direction on an empty board), and let $\mathcal{E}$ be the equivalence relation generated by $\mathcal{R}$.

Here is the formal statement of the exercise: Prove that $\mathcal{E}$ has exactly two equivalence classes, so that the equivalence class of a point is determined by whether $i + j$ is even or odd.

**2.** Suppose that $\mathcal{R}_1$ is an equivalence relation on $X$, let $X/\mathcal{R}_1$ denote the set of equivalence classes for $\mathcal{R}_1$, and let $\mathcal{R}_2$ be an equivalence relation on $X/\mathcal{R}_1$. Define a binary relation $\mathcal{S}$ on $X$ such that $x \ \mathcal{S} \ y$ if and only if the equivalence classes $[x]$ and $[y]$ of $x, y \in X$ with respect to $\mathcal{R}_1$ satisfy $[x] \ \mathcal{R}_2 \ [y]$. Prove that $\mathcal{S}$ also defines an equivalence relation on $X$.

## 3. More on sets and functions

**1.** A set $J$ is called an *initial object* if for each set $X$ there is a unique function $f : J \to X$, and a set $T$ is called a *terminal object* if for each set $X$ there is a unique function $g : X \to T$. Prove that the empty set is the only initial object and the terminal objects are precisely the one point sets of the form $\{p\}$ for some $p$.

**2.** Given two sets $A$ and $B$, their *disjoint union* or *abstract sum* $A \amalg B$ is given by

$$A \amalg B \quad = \quad A \times \{1\} \ \cup \ B \times \{2\} \quad \subset \quad (A \ \cup \ B) \ \times \ \{1, 2\}$$

so that $A \amalg B$ is a union of two disjoint subsets, one of which is in 1–1 correspondence with $A$ and the other of which is in 1–1 correspondence with $B$ (see the comments below regarding the choice of symbols).

($i$) If $C$ is a third set, describe a 1–1 correspondence from $(A \amalg B) \times C$ to $(A \times C) \amalg (B \times C)$. [*Hint:* The left hand side is a subset of $(A \cup B) \times \{1,2\} \times C$, and the right hand side is a subset of $(A \cup B) \times C \times \{1,2\}$.]

($ii$) If $X$ is another set and $f : A \to X$, $g : B \to X$ are functions, prove that there is a unique function $h : A \amalg B \to X$ such that $h(a,1) = f(a)$ for all $a \in A$ and $h(b,2) = g(b)$ for all $b \in B$.

*Remark on the notation.* The symbol $\amalg$ is an upside down upper case Greek Pi (=$\Pi$). One of the reasons for this choice of symbols is that this construction can be viewed as a "dual" to the Cartesian product, which is denoted by $\Pi$, and another is that $\amalg$ is similar but not identical to the usual symbol $\cup$ for the union of two sets.

# 4. Review of some real analysis

Given a sequence $\{a_n\}$ indexed by the nonnegative integers (or all integers greater than or equal to some fixed $N_0$, a *subsequence* of $\{a_n\}$ is a composite construction $\{a_{n(k)}\}$ where $n(k)$ is an integer valued sequence which is strictly increasing as a function of $k$. For example, one can construct the subsequence $\{a_{2n}\}$ of even terms in the original sequence or the subseqnece $\{a_{n^2}\}$. — This concept is used more than once in Sutherland, but it is not defined formally there.

**1.** Suppose that $\{a_n\}$ is a sequence of real numbers which converges to some limit value $L$ in the extended real number system (so $L$ may be $\pm\infty$), and let $\{a_{n(k)}\}$ be a subsequence of $\{a_n\}$. Prove that $\{a_{n(k)}\}$ also converges to $L$.

**2.** ($i$) Prove the real number system has the **Cantor nested intersection property**: If we are given a sequence of closed intervals $\{[a_k, b_k]\}$ in the real numbers such that for each $n$ we have $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$, then these is at least one point $p$ which lies in all the intervals.

($ii$) Suppose that the endpoints in ($i$) are all rational numbers. Does it follow that there is a rational number which lies in all the intervals? Prove this or give a counterexample.