ADDITIONAL EXERCISES FOR

MATHEMATICS 145A — Part 2

Winter 2014

5. Metric spaces

1. Let (X,d) be a metric space, and let $U \subset X$ be an open subset. For each $u \in U$ choose $\delta(x) > 0$ such that the open neighborhood $N_{\delta(x)}(x)$ of radius $\delta(x)$ with center x is contained in U. Prove that

$$U \quad = \quad \bigcup_{x\in U} N_{\delta(x)}(x) \ .$$

2. Let (X,d) be a metric space such that X is a finite set. Prove that every subset is open. [*Hint:* Why does it suffice to prove that every one point subset is open? Explain why the set of positive distances between two points has a positive lower bound.]

3. A function on metric spaces $f : (X,d^X) \to (Y,d^Y)$ is an *open mapping* if for each open set $U \subset X$ the image $f[U]$ is open in Y. Prove that f is an open mapping if and only if for each open subset $U \subset X$ and $x \in U$ there is some open set W_x in X such that $x \in W_x \subset U$ and $f[W_x]$ is open in Y. [*Hint:* Verify that $U = \cup_{x\in U} W_x$.]

4. Let (X,d^X) and (Y,d^Y) be metric spaces, and let $f : X \to Y$ be a 1–1 correspondence, and let h be the inverse function to f. Prove that f is continuous if and only if h is an open mapping.

5. A metric space (X,d) is said to be an *ultra-metric space* if $d(x,z) \leq \max\{d(x,z),\ d(y,z)\}$ for all $x,y,z \in X$. Show that if d is the standard discrete metric then (X,d) is ultra-metric, and that $\mathbb{R}$ with the usual metric is not.

6. This exercise deals with some properties of subsets in $\mathbb{R}^n$ known as *(piecewise) self-similar fractals*. — More generally, if (X,d) is a metric space and $A \subset X$ is a bounded subset, then A is said to be *piecewise self-similar* if A is a union of finitely many subsets $A_1, \ \cdots\ , A_m$ such that for each $k = 1, ..., m$ there is a 1–1 correspondence $g_k : A \to A_k$ such that

$$d\big(\, g_k(a),\ g_k(a')\,\big) \quad = \quad r \cdot d(a,a')$$

for some r satisfying $0 < r < 1$ which is the same for each choice of k.

In most books and papers the modifying term "piecewise" is omitted, but we include it here because it provides a more accurate description of these objects and the hypothesis in (iv) seems like a more traditional and direct way of defining self-similarity.

(i) Show that a closed interval $[0,1]$ piecewise self-similar. [*Hint:* Cut it in half.]

(ii) Show that the Cantor set (see `intro2topA-04.pdf`) is piecewise self-similar. It will be enough to construct the maps g_k; the details of verifying that these maps are 1–1 and onto may be omitted. [*Hint:* The simplest choices for m and r are 2 and 3 respectively.]

(iii) Suppose that A and B are piecewise self-similar subsets of X and Y respectively with the same ratio r in each case (for example, this condition is satisfied if $A = B$ and $X = Y$). Prove that

$A \times B$ is a piecewise self-similar subset of $X \times Y$, where the latter has the d_2 product metric. — In particular, this implies that the hypercube

$$[0,1] \;\times\; \cdots \;\times\; [0,1] \qquad (n \text{ factors})$$

a piecewise self-similar subset of$\mathbb{R}^n$, and likewise for a product of n copies of the Cantor set with itself.

(iv) Suppose that A is a subset of a metric space X and $f : A \to A$ is a 1–1 onto self-map such that

$$d\big(\, f(a),\, f(a')\,\big) \quad = \quad r \cdot d(a, a')$$

for some fixed $r > 0$ and all $a, a' \in A$. Prove that either $r = 1$ or A is not bounded. [*Hint:* If A is bounded and Δ is the diameter of A, why do the hypotheses imply that $r \cdot \Delta = \Delta$?]

(v) Give an example of a bounded subset $A \subset \mathbb{R}$ for which one can find a mapping f satisfying the conditions in (iv) such that $r = 1$ but f is not the identity mapping of A.

Some additional information, presented at the level of this course, appears on pages 7–8 and 33–34 in the following textbook:

> **S. E. Goodman**, *Beginning Topology*, American Mathematical Society, Providence, RI, 2009.

7. Let $[a, b]$ be a closed interval in the real line, let (Y, d^Y) be a metric spac, and let $f : [a, b] \to Y$ be a continuous function. Prove that f can be extended to a continuous function $F : [a, b] \to \mathbb{R}$. Specifically, if we define F so that $F(x) = f(a)$ for $x \leq a$ and $F(x) = f(b)$ for $x \geq b$, verify that F is continuous.

6. Concepts in metric spaces

1. (i) Let (X, d) be a metric space, let $p \in X$, and let $r \geq 0$. Prove that the subset $\{y \in X \mid d(y, x) \;>\; r\}$ is is open in X. [*Hint:* If $y \neq x$ consider $N_\varepsilon(y)$, where $\varepsilon = d^X(x, y) - r$.]

(ii) In the notation of (i), prove that $\{y \in X \mid d(y, x) \;\leq\; r\}$ is closed in X.

2. Let (X, d) be a metric space, and let $A \subset B$ be subsets of X, and let $\mathbf{Clos}\,(A, Y)$ denote the closure of A in Y, where $A \subset Y \subset X$ and Y has the subspace metric. Prove that $\mathbf{Clos}\,(A, B) = \mathbf{Clos}\,(A, X) \cap B$.

3. Let (X, d) be a metric space such that an arbitrary intersection of open subsets is open. Prove that every subset of X is open.

4. Let (X, d^X) and (Y, d^Y) be metric spaces, and let $b \in Y$. Prove that the map $X \to X \times \{b\}$ sending x to (x, b) defines an isometry from (X, d) to $X \times \{b\}$ with the metric indued from the d_p metric with p equal to 1, 2 or ∞ (each metric induces the same subspace metric on $X \times \{b\}$). Formulate and prove a corresonding result involving Y and $\{a\} \times Y$ where $a \in X$.

5. (i) Let (X, d) be a metric space, and let $p \in X$. Prove that $X - \{p\}$ is dense in X if and only if $\{p\}$ is open in X.

(ii) If U and V are open and dense subsets of a metric space (X, d), prove that $U \cap V$ is dense in X (hence a finite intersection of such subsets is also dense by finite induction).

(iii) Prove that a countable intersection of open dense subsets in (X, d) is not necessarily dense. [*Hint:* Let $X = \mathbb{Q}$ with the usual metric.]

6. A *hyperplane* in $\mathbb{R}^n$ is a subset of $\mathbb{R}6n$ defined by a nontrivial first degree polynomial equation

$$0 = F(x_1, \cdots, x_n) = \left(\sum_{i=0}^{n} a_i x_i\right) - b$$

where a_i and b are real numbers and at least one of the coefficients a_i is nonzero. If H is such a hyperplane, prove that its interior is empty and $\mathbb{R}^n - H$ is dense in $\mathbb{R}^n$. [*Hint:* If N is the normal vector to H with coordinates $a = (a_1, \cdots, a_n)$ prove that the function $F(y + ta)$ is not constant. Why does this imply that if $y \in H$ then $y + ta \notin H$ if $t \neq 0$?]

Note. We can combine the conclusions of the preceding two exercises with an induction argument to conclude that the complement of a union of finitely many hyperplanes is open and dense in $\mathbb{R}^n$.

7. (i) Using Exercise 6.16 in Sutherland, prove that every closed subset of a metric space is a countable intersection of open subsets (a set of this type is often called a G_δ set). [*Hint:* If $A \subset X$ and X is a metric space and $\varepsilon > 0$, show that the set of all x such that $d(x, A) < \varepsilon$ is open.]

(ii) Prove that every open subset of a metric space is a countable union of closed subsets (a set of this type is often called an F_σ set).

8. Let (X, d) be a metric space, and let C and Y be subsets of X. Prove the relative boundary relationship

$$\mathrm{Bdy}\,(C \cap Y, Y) \quad \subset \quad \mathrm{Bdy}\,(C, X)$$

where $\mathrm{Bdy}\,(A, B)$ denotes the boundary of A in B for $A \subset B \subset X$. Give an example where $X = \mathbb{R}$ with the usual metric and the containment is proper.

9. Let $a < b$ in $\mathbb{R}$, let f and g be continuous functions on $[a, b]$ such that $g(x) < f(x)$ for all $x \in (a, b)$ and $g(x) \leq f(x)$ for $x = a, b$. Define $V, A \subset \mathbb{R}^2$ as follows:

$$A = \{(x, y) \mid a \leq x \leq b\,, \quad g(x) \leq y \leq f(x)\}$$

$$V = \{(x, y) \mid a < x < b\,, \quad g(x) < y < f(x)\}$$

Results in `math145Anotes06.pdf` show that V and A are (respectively) open and closed subsets of $\mathbb{R}^2$. Prove that A is the closure of U and U is the interior of A. Also, prove that the boundary of each subset in $\mathbb{R}^2$ is the union of the following four curves:

C_1: The graph $y = g(x)$ where $a \leq x \leq b$.

C_2: The vertical line segment $x = b$ with $g(b) \leq y \leq f(b)$.

C_3: The graph $y = f(x)$ where $a \leq x \leq b$.

C_4: The vertical line segment $x = a$ with $g(a) \leq y \leq f(a)$.

Hints and drawings for this problem are contained in the file `solutions02w14.figures.pdf`.

Generalizations. There are analogous results for the standard types of domains in $\mathbb{R}^3$ over which one takes triple integrals, which are defined by the following inequalities such that all functions in the displayed lines are continuous:

$$a \quad \leq \quad x \quad \leq \quad b \ , \qquad u(x) \quad \leq \quad y \quad \leq \quad v(x) \ , \qquad f(x,y) \quad \leq \quad z \quad \leq \quad g(x,y)$$

One can formulate and prove analogs of Exercise 9 for this closed set and the corresponding open set in which the inequalities are strict (for the sake of simplicity let's assume that $u < v$ and $f < g$ on the interval $[a, b]$ and the closed domain

$$\{(x,y) \ \mid \ a \quad \leq x \quad \leq \quad b \ , \quad u(x) \quad \leq \quad y \quad \leq \quad v(x) \ \}$$

respectively). In this case the boundary has six pieces to consider (just like the special case of a rectangular region defined by $a \leq x \leq b$, $c \leq y \leq d$ and $p \leq z \leq q$) and the bookkeeping involving the boundary will be more complicated, so we shall not attempt to give precise statements or proofs. There are also similar generalizations for suitably defined subsets of $\mathbb{R}^n$ where $n \geq 4$, but we shall not attempt to describe them here.