ADDITIONAL EXERCISES FOR

MATHEMATICS 145A — Part 3

Winter 2014

7. Topological spaces

1. Let $\mathcal{U}$ denote the topology in Exercise 7.6 of Sutherland, which consists of $\mathbb{R}$, the empty set and all intervals $(-\infty, b)$ where b is some real number. If $(X, \mathcal{T})$ is a topological space, a continuous function $f : (X, \mathcal{T}) \to (\mathbb{R}, \mathcal{U})$ is said to be *upper semi-continuous*.

(i) Let $(X, \mathcal{T})$ be a topological space, and let f be a real valued function on X. Prove that f is upper semi-continuous if and only if for each $x \in X$ and $\varepsilon > 0$ there is some open set $U_{\varepsilon,x}$ containing x such that $y \in U_{\varepsilon,x}$ implies $f(y) < f(x) + \varepsilon$.

(ii) Let $[a, b]$ be a closed interval in $\mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$ be the function whose value is 1 on $[a, b]$ and zero elsewhere, and let d be the usual metric on $\mathbb{R}$. Prove that $f : (\mathbb{R}, d) \to (\mathbb{R}, \mathcal{U})$ is upper semi-continuous.

(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function (but not necessarily strictly increasing), and let d be the usual metric on $\mathbb{R}$. Prove that if f satisfies the one-sided continuity condition $f(x) = \text{G.L.B.}_{t>x}\, f(t)$ for all x, then $f : (\mathbb{R}, d) \to (\mathbb{R}, \mathcal{U})$ is upper semi-continuous.

(iv) Let $a, b \in \mathbb{R}$ with $a \geq 0$, and let $f(x) = ax+b$, where $x \in \mathbb{R}$. Prove that $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{U})$ is continuous.

(v) Let $a, b \in \mathbb{R}$ with $a \geq 0$, and let $f(x) = -x$, where $x \in \mathbb{R}$. Prove that $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{U})$ is not continuous.

2. In the topological space $(\mathbb{R}, \mathcal{U})$ from the preceding exercise, prove the following:

(i) Given $x \neq y$, there is an open set V containing one of these points but not the other.

(ii) If $x < y$, then every open set which contains y also contains x. Deduce from this that for each $x \in \mathbb{R}$ the subset $\mathbb{R} - \{x\}$ is not open with respect to $\mathcal{U}$.

8. Continuity in topological spaces; bases

1. If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, a function $f : X \to Y$ is said to be an *open mapping* if for each open subset $U \subset X$, the image $f[U]$ is open in Y, one can define a *closed mapping* similarly, replacing "open" with "closed" everywhere. — Prove that the composite of two open mappings is open, and the composite of two closed mappings is closed.

2. Let X be a set, let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X, and let $j_X : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ be the identity map. Prove that j_X is continuous if and only if $\mathcal{T}_2$ is contained in $\mathcal{T}_1$, and j_X is open if and only if $\mathcal{T}_1$ is contained in $\mathcal{T}_2$.

3. Show that the function below defines a homeomorphism of $\mathbb{R}^2$ by describing the inverse explicitly:

$$F(x,y) \quad = \quad \left(xe^y + y \ , \ xe^y - y\right)$$

4. Show that the function below defines a homeomorphism of $\mathbb{R}^3$ by describing the inverse explicitly:

$$F(x,y,z) \quad = \quad \left(\frac{x}{2+y^2} + ye^z \ , \ \frac{x}{2+y^2} - ye^z \ , \ 2ye^z + z\right)$$

5. (i) Show that the function $f(x) = e^x + x$ is strictly increasing and satisfies

$$\lim_{x \to -\infty} f(x) \quad = \quad -\infty \ , \qquad \lim_{x \to +\infty} f(x) \quad = \quad +\infty$$

and using the Intermediate Value Theorem explain why g has a continuous inverse. [*Hint:* Use a deriviative test to show g is increasing.]

(ii) Show that the function below defines a homeomorphism of $\mathbb{R}^2$:

$$F(x,y) \quad = \quad \left(e^x + y \ , \ x - y\right)$$

[*Hint:* The formula for the inverse to F will use the inverse to the function g considered in part (i).]

Note. A general version of the Intermediate Value Theorem is stated and proved in Chapter 12 of Sutherland; for the purposes of working part (i) of Exercise 5, the validity of the theorem can be assumed.