ADDITIONAL EXERCISES FOR
MATHEMATICS 145A — Part 4

Winter 2014

9. Some concepts in topological spaces

1. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and that $\mathcal{V}$ is a family of subsets which generates $\mathcal{T}_Y$. Prove that a function $f : X \to Y$ is continuous if and only if for each $V \in \mathcal{V}$ the inverse image $f^{-1}[V]$ is open in X.

2. Let $D^2 \subset \mathbb{R}^2$ be the set of all v such that $|v| \leq 1$, and let $N_1(0) \subset \mathbb{R}^2$ be the subset defined by $|v| < 1$. Prove that the boundary of each of these subsets is the unit circle S^1 defined by the equation $|v| = 1$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is continuous with respect to the usual topology on $\mathbb{R}$, and define the **graph** of f to be the subset Γ_f consisting of all $(x, y) \in \mathbb{R}^2$ such that $y = f(x)$. Prove that Γ_f is the boundary for each of the following subsets:

$$\{(x,y) \in \mathbb{R}^2 \mid y < f(x)\}, \qquad \{(x,y) \in \mathbb{R}^2 \mid y > f(x)\}$$

[*Hint:* Look back at a previous additional exercise for Chapter 6.]

10. Subspaces and product spaces

1. Suppose that X is a topological space and $A \subset B \subset X$. If A is dense in B (with respect to the subspace topology on B) and B is dense in X, prove that A is dense in X.

2. Given topological spaces X and Y, suppose that $X \times Y$ has the product topology, and let π_X and π_Y denote the coordinae projections onto X and Y respectively. Prove that these two mappings (which are continuous and open) are not necessarily closed. [*Hint:* Look at the graph of $f(x) = 1/x$ for $x \neq 0$.]

3. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, and assume that A and B are subsets of X and Y respectively. If $\mathcal{T}_X \prod \mathcal{T}_Y$ denotes the product topology and $\mathcal{T}_X|A$, $\mathcal{T}_Y|B$ denote the respective subspace topologies, prove that

$$(\mathcal{T}_X \prod \mathcal{T}_Y) \mid A \times B \quad = \quad (\mathcal{T}_X \mid A) \prod (\mathcal{T}_Y) \mid B .$$

In words, *a topological product of subspaces is a subspace of the topological product*. [*Hint:* Show that both topologies are generated by the same subsets of $A \times B$.]

4. (i) Suppose that $(X, \mathcal{T}_X)$ is a topological space and $f : X \to Y$ is a function with values in some set Y. Prove that there is a unique maximal topology $f_*\mathcal{T}_X$ on Y such that f is continuous. [*Hint:* If f is continuous, what is the largest family of subsets in Y whose inverse images could be open?]

(ii) Suppose that $(Y, \mathcal{T}_Y)$ is a topological space and $f : X \to Y$ is a function defined on some set X. Prove that there is a unique minimal topology $f^*\mathcal{T}_Y$ on X such that f is continuous. [*Hint:* If f is continuous, what is the smallest family of subsets in X whose inverse images must be open?]

Note. The topologies in the preceding exercise are sometimes called the *co-induced topology* and the *induced topology* respectively.