Supplement to Chapter 7 of Sutherland,

Introduction to Metric and Topological Spaces (Second Edition)

Different topologies on the same set

The examples in Chapter 7 of Sutherland show that there are many topological structures with the same underlying set. In fact, there are situations where this is more than just a formal curiosity; usually one has two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on an underlying set $\mathbf{X}$ and every open set in one of the topologies (say, $\mathcal{T}_1$) is also open in the other. Sutherland formalizes this notion, saying that in such cases the topology $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$, and conversely the topology $\mathcal{T}_2$ is finer than $\mathcal{T}_1$. Some of the most obvious examples of this sort arise when $\mathbf{X}$ is $\mathbb{R}^n$, with $\mathcal{T}_1$ equal to the usual topology and $\mathcal{T}_2$ equal to the discrete topology.

Probably the most immediately useful examples involving pairs of topologies $\mathcal{T}_1 \subset \mathcal{T}_2$ arise in the study of ***normed vector spaces***, which are defined near the top of page 40 in Sutherland. For infinite – dimensional normed vector spaces, one can define a ***weak topology*** $\mathcal{T}_1$ which is properly contained in the usual metric topology $\mathcal{T}_2$ and turns out to be useful for many purposes in functional analysis. It is beyond the scope of this course to discuss this in more detail, but here is one introductory online reference.

http://en.wikipedia.org/wiki/Weak_topology

Warning. Teminology for pairs of topologies is not consistent in the literature, and **there are papers and books where coarser and finer, and stronger and weaker, have the *exact opposite* meanings from those stated above.** The only reliably consistent terms for comparing topology are that one is larger or smaller than the other.