Fourth Supplement to Chapter 12 of Sutherland,

Introduction to Metric and Topological Spaces (Second Edition)

The starting point is the following result, which was discussed at the end of Chapter 8 in `math145Anotes07-09.pdf`:

PROPOSITION. *Let $a < x < y < b$ in $\mathbb{R}$. Then there is a homeomorphism f from (a, b) to itself such that f is strictly increasing and $f(x) = y$. In fact, f extends to a homeomorphism from $[a, b]$ to itself with $f(a) = a$ and $f(b) = b$.*

COROLLARY. *If $a < b \in \mathbb{R}$ and $u, v \in (u, v)$, then there is a homeomorphism f from (a, b) to itself such that f is strictly increasing and $f(u) = v$. In fact, f extends to a homeomorphism from $[a, b]$ to itself with $f(a) = a$ and $f(b) = b$.*

Derivation of the corollary. If $u < v$ this is the conclusion of the proposition, and if $u = v$ we can take f to be the identity. In the remaining case where $v < u$, we can use the proposition to find a homeomorphism g from (a, b) to itself such that f is strictly increasing and $g(v) = u$ (and g extends to $[a, b]$ as in the proposition). If $f = g^{-1}$, then f has all the required properties.∎

Proof of the proposition. The underlying geometric idea is simple; namely, we stretch the interval $[a, x]$ linearly to $[a, y]$ and shrink the interval $[x, b]$ linearly to $[y, b]$. If we define linear functions $g_-(t)$ and $g_+(t)$ as below, the first function will do the desired stretching and the second will do the desired shrinking:

$$g_-(t) \quad = \quad a \;+\; (t - a) \cdot \frac{y - a}{x - a} \;, \qquad g_+(t) \quad = \quad b \;+\; (t - b) \cdot \frac{y - b}{x - b}$$

By construction the functions g_- and g_+ are strictly increasing linear functions (the coefficient of t is positive in each case), and we also have

$$g_-(a) \;=\; a \;, \qquad g_-(x) \;=\; y \;, \qquad g_+(x) \;=\; y \;, \qquad g_+(b) \;=\; b \;.$$

If we now define f on $[a, b]$ by $f(t) = g_-(t)$ for $t \leq x$ and $f(t) = g_+(t)$ for $t \geq x$, then we obtain a well-defined and strictly increasing continuous function which maps $[a, b]$ to itself and sends each end point to itself. We can now use the main result in `math145Anotes12a.pdf` to conclude that f has a continuous inverse and hence is a homeomorphism.∎

The focus of this document is on generalizing the proposition to connected open subsets in higher dimensional coordinate (= Euclidean) spaces $\mathbb{R}^n$. We shall concentrate on the case $n = 2$, and at the end we shall indicate how one can prove a generalization to higher values of n.

We shall use the following result/example from `intro2topA-08.pdf` (see page 5 of that document):

EXAMPLE. *There is a homeomorphism of the solid square $[-1, 1] \times [-1, 1]$ to itself with the following properties:*

(1) *The center $(0, 0)$ is mapped to the point $(a, 0)$ where a is an arbitrary point in the open unit interval $(0, 1)$.*

(2) *Every point on the boundary of the square, which is equal to*

$$\{-1,1\} \times [-1,1] \quad \cup \quad [-1,1] \times \{-1,1\}$$

is sent to itself (by the identity map).

(3) *The t-axis is sent to the broken line curve with two linear segments, one joining $(0,-1)$ to $(a,0)$ and the other joining $(a,0)$ to $(0,1)$.*

(4) *The square regions formed by intersecting the square with the four closed quadrants (defined by $\{x,y \geq 0\}$, $\{x \geq 0 \geq y\}$, $\{0 \geq x,y\}$ and $\{y \geq 0 \geq x\}$ respectively) are sent to closed regions bounded by trapezoids (there is a drawing on page 5 of the document cited above).*

Furthermore, an explicit formula for this homeomorphism is given by

$$F(s,t) \quad = \quad \big(a - a\,|t| + s(1 - a + a\,|t|),\, t\big) \qquad \text{if} \quad s \ \geq \ 0$$

$$F(s,t) \quad = \quad \big(a - a\,|t| + s(1 + a - a\,|t|),\, t\big) \qquad \text{if} \quad s \ \leq \ 0$$

One can check directly that the two formulas yield the same value at the points where $s \geq 0$ and $s \leq 0$ (*i.e.*, $s = 0$).

We shall use the construction described above to prove the following result on homeomorphisms of a disk:

CONSTRUCTION LEMMA. *Let $\varepsilon > 0$, let $\varepsilon\, D$ denote the closed disk defined by $x^2+y^2 \leq \varepsilon^2$, and suppose that $v \in \mathbb{R}^2$ satisfies $|v| < \varepsilon/\sqrt{2}$. Then there is a homeomorphism h from $\varepsilon\, D^2$ to itself such that h is the identity on the boundary circle (defined by $x^2 + y^2 = \varepsilon^2$) and $h(0) = v$.*

Proof. The first step is to prove this result when $\varepsilon = \sqrt{2}$ and v is a positive multiple of the first unit vector such that $|v| < 1$. Then $\sqrt{2}\, D$ is the union of the square $E = [-1,1] \times [-1,1]$ and the set C defined by $x^2 + y^2 \leq 2$ and either $|x| \geq 1$ or $|y| \geq 1$ (or both). The intersection of these sets is the boundary of the square described in (2) above. Define h on $\sqrt{2}\, D$ such that h is the previously described homeomorphism F on E and h is the identity on C. These functions combine to form a well-defined continuous function on $\sqrt{2}\, D$ because the restriction of F to the boundary of E is the identity (see `intro2topA-10b.pdf` for more on this point). In fact, the mapping h constructed by this procedure is a homeomorphism, for we can construct a continuous inverse explicitly by taking F^{-1} on E and the identity on C (the discussion in the last paragraph on page 5 of the file `intro2topA-08.pdf` can be used to obtain a formula for F^{-1} and show that the latter is continuous). It follows immediately that this homeomorphism has all the required properties for the given choice of v.

The second step is to prove the result when $\varepsilon = \sqrt{2}$ and v is an arbitrary vector such such that $0 < |v| < 1$. Let $\mathbf{u}$ be the unit vector $|v|^{-1} \cdot v$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear rotation sending the first unit vector $(1,0)$ into $\mathbf{u}$. If h is the homeomorphism in the first paragraph which sends $(0,0)$ to $(|v|,0)$, then $T \circ h \circ T^{-1}$ will be the desired homeomorphism from $\sqrt{2}\, D$ to itself which is the identity on the boundary and sends $(0,0)$ to v.

The final step is to prove the result when $\varepsilon > 0$ is arbitrary and v is a nonzero vector such that $|v| < \varepsilon/\sqrt{2}$. Let k be the homeomorphism constructed in the preceding step such that

$$k(0,0) \quad = \quad \frac{\sqrt{2}}{\varepsilon} \cdot v$$

and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the scalar multiplication homeomorphism defined by

$$S(a) \quad = \quad \frac{\varepsilon}{\sqrt{2}} \cdot a$$

for all $a \in \mathbb{R}^2$. Then $S \circ h \circ S^{-1}$ will be the desired homeomorphism from $\varepsilon\, D$ to itself which is the identity on the boundary and sends $(0, 0)$ to v.■

We are now ready to prove the main result:

THEOREM. *Let U be a connected open subset of $\mathbb{R}^2$. Then for each pair of points $x, y \in U$ there is a homeomorphism h from U to itself such that $h(x) = y$.*

Proof. For each $u \in \mathbb{R}^2$ let T_u be the translation map $T_u(w) = u + w$. Suppose now that $x \in U$, and choose $\varepsilon > 0$ so that $N_{2\varepsilon}(x) \subset U$. If Δ is the closed disk of radius ε centered at x and $y \in \Delta$ satisfies $|y - x| < \varepsilon/\sqrt{2}$, let φ be the homeomorphism in the Construction Lemma from $\varepsilon\, D$ to itself such that φ is the identity on the boundary and $\varphi(0) = y - x$. Then $T_x \circ \varphi \circ T_x^{-1}$ defines a homeomorphism from Δ to itself which is again the identity on the boundary and sends x to y. We can extend φ to a homeomorphism Φ on all of U by setting Φ equal to the identity on $U - N_\varepsilon(x)$; since φ is equal to the identity on the boundary of Δ, it follows that Φ is a well-defined homeomorphism from U to itself such that $\Phi(x) = y$.

Now define a binary relation on the open set U by $p \sim q$ if and only if there is a homeomorphism h from U to itself sending p to q; it follows immediately that $\sim$ is an equivalence relation. Furthermore, the construction of the preceding paragraph shows that each equivalence class of U is open. Since the complement of an equivalence class is a union of the other equivalence classes, it follows that an equivalence class is also closed in U. Since U is connected, there can be only one equivalence class, and this yields the conclusion of the theorem.■

Generalization to higher dimensions

We can prove a similar result for connected open subsets in $\mathbb{R}^n$ when $n \geq 3$. This will only require some relatively small adjustments to the preceding argument(s). In the construction of a homeomorphism from $[-1, 1] \times [-1, 1]$ to itself from `intro2topA-08.pdf`, everything will go through if we replace the second factor in the product, with the closed unit disk D^{n-1} consisting of all points $t \in \mathbb{R}^{n-1}$ such that $|t| \leq 1$ (note that $[-1, 1] = D^1$). The boundary of the solid cylindrical region $C_n = [-1, 1] \times D^{n-1}$ is then equal to

$$\{-1, 1\} \times D^{n-1} \quad \cup \quad [-1, 1] \times S^{n-2}$$

(where the unit sphere S^{n-2} is defined by $|t| = 1$), the constructed homeomorphism is the identity on this boundary, and C_n itself is contained in the disk $\sqrt{2}\, D^n \subset \mathbb{R}^n$ of all points z such that $|z| \leq \sqrt{2}$. As in the first step of the Construction Lemma, we can extend the homeomorphism on $[-1, 1] \times D^{n-1}$ to a homeomorphism on all of $\sqrt{2}\, D^n$ by taking the extension to be the identity off the open cylinder defined by the strict inequalities $|s| < 1$ and $|t| < 1$.

We can now proceed as in the second step of the Construction Lemma. Given such that $0 < |v| < 1$, let $\mathbf{u}$ be the unit vector $|v|^{-1} \cdot v$, and replace the linear rotation T of $\mathbb{R}^2$ by an orthogonal linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ sending the first unit vector $(1, 0, \cdots, 0)$ into $\mathbf{u}$. With this adjustment, the second step in the proof of the Construction Lemma goes through without further changes. This in turn allows us to use the same argument as in the final step to prove an extension of the Construction Lemma in which $\mathbb{R}^2$ is replaced by $\mathbb{R}^n$, where $n \geq 2$ is arbitrary.

The preceding conclusions imply that if U is an open connected subset in $\mathbb{R}^n$ (where $n \geq 2$) and $x \in U$, then there is some $\delta > 0$ such that $N_\delta(x) \subset U$ and for each $y \in N_\delta(x)$ there is a homeomorphism h from U to itself such that $h(x) = y$; the case $n = 2$ is established in the first paragraph of the proof of the theorem, and we can use the generalization of the Construction Lemma 1 to prove a similar result for all $n \geq 2$.

Finally, we can complete the proof of the generalized theorem as follows: The equivalence relation $\sim$ can be defined on a connected open subset of any topological space, and the reasoning of the preceding paragraph implies that the equivalence classes of $\sim$ are open if U is an open subset of $\mathbb{R}^n$ for $n \geq 2$. From this point on, the argument in the proof of the theorem goes through without further adjustments.$\blacksquare$