1. INTRODUCTION

One goal — to construct a general framework for discussing concepts like regions and continuous functions.

Concrete setting $\mathbb{R}^n$, functions of several real variables.

Two levels of abstraction

Metric spaces — abstract notions of distance

Topological spaces — abstract regions (open)

Sometimes it is convenient to have more data, sometimes it is convenient to have less.

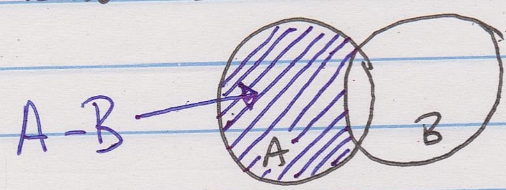
One benefit of the abstraction is that it ~~leads~~ reveals ~~to~~ new and important insights. In particular, some concern ways in which two geometrical objects resemble — or do not resemble — each other.

Prerequisites here are less than in Sutherland — only set theory (144) and multivariable calculus (as in 10A and 10B).

2. NOTATION & TERMINOLOGY

A course in set theory is a prerequisite. Set theoretic notation and vocabulary will be used freely without explanations. (This includes unions, intersections, ...)

Notation for relative complements: $A-B$ (Sutherland uses $A \setminus B$).



Standard notation for intervals in reals ($\mathbb{R}$)

$[a,b]$ CLOSED CLOSED $\quad (a,b)$ OPEN $\quad \underbrace{[a,b) \quad (a,b]}_{\text{HALF-OPEN}}$

At the open ends we allow values of $\pm\infty$.

Functions (maps, mappings) (or morphisms)

Formally, these are triples
$f = (A, \Gamma_f, B)$ $\quad \Gamma_f \subseteq A \times B$ such that for each $a \in A$ there is a unique $b \in B$ such that $(a,b) \in \Gamma_f$. Write $b = f(a)$ if $(a,b) \in \Gamma_f$.

Note that $A \times B$ = all ordered pairs (a, b) such that $a \in A$ & $b \in B$.

> Given two objects a & b one can form an ordered pair (a, b) such that $(a, b) = (c, d) \iff a = c$ & $b = d$

Write $f : A \longrightarrow B$ and say that A is the <u>domain</u> of f, B is the <u>co-domain</u> of f.
(Not everyone specifies codomains explicitly, but ultimately this is necessary in many contexts. — There are analogies in computer languages, where one must specify whether the values of certain functions are whole numbers or decimal expressions.)

<u>Synonyms for special types of functions</u>

1-1 into
injective
monomorphism

onto
surjective
epimorphism

(BOTH VALID)

1-1 onto or 1-1 correspondence
bijective
isomorphism

3. More on Sets & Functions

Main concepts $\qquad f: A \to B$ function

also sometimes $A \xrightarrow{f} B$

(1) If $C \subseteq A$, the image of C under f.

(2) If $D \subseteq B$, the inverse image of D with respect to f.

(3) Inverse functions.

All play ~~an~~ important roles in the course.

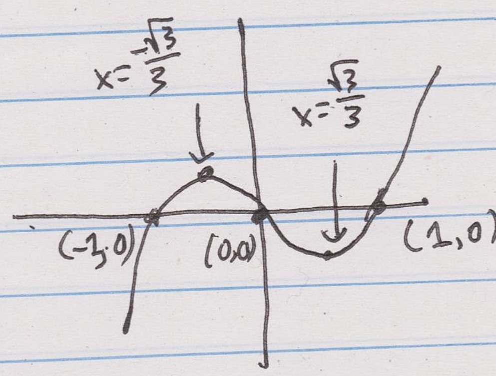
(1) Images

$$f[C] = \text{all } b \in B \text{ such that } b = f(c), \text{ some } c \in C.$$

(2) $f^{-1}[D] = \text{all } a \in A \text{ such that } f(a) \in D.$

Two examples are on p. 10 of Sutherland.
Here is one more.

$$f(x) = x^3 - x.$$



$f: \mathbb{R} \to \mathbb{R}$

Suppose $C = [0,1]$. Then

$$f[C] = \left[\left(\frac{\sqrt{3}}{3}\right)^3 - \left(\frac{\sqrt{3}}{3}\right), 0\right].$$

{NO PROOFS!}

↑
minimum value.

Suppose $D = (-\infty, 0]$.
Then $f^{-1}[D] = (-\infty, -1] \cup [0,1]$.

<u>Yet another example</u> $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{1}{x^2+1}.$$

(since $0 < f(x) < 1$)

$f[\mathbb{R}] = (0, 1]$

$f^{-1}[D] = \mathbb{R}$ if $D = [0,1]$ or $[0,2]$

$f^{-1}[D] = \phi$ if $D = [-1, 0]$.

If $C = [0,1]$, then $f[C] = [\frac{1}{2}, 1]$.

Various identities are on pp. 10-13 of Sutherland. For example

$f[C_1 \cup C_2] = f[C_1] \cup f[C_2]$

$f^{-1}[D_1 \cup D_2] = f^{-1}[D_1] \cup f^{-1}[D_2]$

$f^{-1}[D_1 \cap D_2] = f^{-1}[D_1] \cap f^{-1}[D_2]$.

However, we only have
$$f[C_1 \cap C_2] \subseteq f[C_1] \cap f[C_2].$$

Example $C_1 = [-2,-1]$ $C_2 = [1,2]$

$f(x) = x^2$. $C_1 \cap C_2 = \phi \Rightarrow LHS = \phi$

but $RHS = [1,4]$.

Potential source of confusion !!

The terminology $f^{-1}[D]$ does $\boxed{\text{not}}$ mean that there is an inverse function f^{-1}.

Two characterizations of functions with inverses (invertible functions): or is it three?

An inverse function $g: Y \to X$ to $f: X \to Y$ has the property that $x = g(y) \Leftrightarrow y = f(x)$.

(1) If f is 1-1 onto, then one has such a function because for each $y \in Y$ there is exactly one $x \in X$ such that $f(x) = y$.

IMPORTANT !!

(2) Suppose that there is $h: Y \to X$ such that $h(f(x)) = x$ all x and $f(h(y)) = y$ all y. Then f is 1-1 onto and h is an inverse.

<u>Proof of (2)</u> <u>f is 1-1</u> $f(x) = f(x') \Rightarrow$

$x = h(f(x)) = h(f(x')) = x'$

<u>f is onto</u> $y = f(h(y))$ for all $y \in Y$.

<u>inverse identities</u> $y = f(x) \Rightarrow x = h(f(x)) = h(y)$.

$x = h(y) \Rightarrow f(x) = f(h(y)) = y$. ■

BIG DOT = END OF ARGUMENT.

We often, but not always, write $h = f^{-1}$

<u>Sutherland, Proposition 3.20, rewritten</u>

$f: X \to Y$ 1-1 onto, $h =$ inverse fcn.

$V \subseteq X$. Then $f[V] = h^{-1}[V]$.

<u>Proof.</u> $y = f(v)$ for some $v \in V \Leftrightarrow$

$v = h(y)$. The $(\Rightarrow)$ implication shows that $f[V] \subseteq h^{-1}[V]$, and the $(\Leftarrow)$ implication shows that $h^{-1}[V] \subseteq f[V]$. ■

<u>NOTE</u> We shall also observe the convention in the next to last ¶ on p. 14 of Sutherland.