5. Metric Spaces

Problem. Find the minimal data needed to establish the results on continuous functions in the previous chapter.

In particular, we want a setting which includes continuous functions defined on various subsets of coordinate n-space $\mathbb{R}^n$.

A few simply stated properties of distance turn out to be sufficient for extending a large class of concepts and results about regions in $\mathbb{R}^n$ and continuous functions.

"a lot of our geometric intuition" carries over to this setting
(Sutherland, ¶ before Prop 5.2, p.39)

Definition A *metric space* (X, d) is a pair consisting of a set X (the space) and a function

$$\cancel{d} \quad d : X \times X \longrightarrow \mathbb{R}$$

(the *metric* or *distance function*)

such that the following hold:

(M1) $d(x,y) \geq 0$, equality $\Leftrightarrow x=y$.

(M2) $d(x,y) = d(y,x)$

(M3) $d(x,z) \leq d(x,y) + d(y,z)$

all x, y, z in X

We often write just X if d is clear in context or more information is unneeded

Results on inner products imply that $\mathbb{R}^n$ is a metric space with $d(x,y) = \|x-y\|$ vector length

$= \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$ Pythagorean Metric

There are many others in Sutherland.

One family

Normed vector spaces

$(V, |\cdot\cdot|)$ V = real vector space

$|v|$ = norm of $v \in V$, $|\ldots| : V \to \mathbb{R}$ s.t.

$$\left.\begin{array}{l} |v| \geq 0, \text{ equality} \Leftrightarrow v = 0 \\ |cv| = |c| \cdot |v| \\ |v_1 + v_2| \leq |v_1| + |v_2| \end{array}\right\} \begin{array}{l} \text{all} \\ v, c \\ v_1, v_2 \end{array}$$

c=scalar

Verifications that $d(x,y) = |x-y|$ defines a metric.

(M1) $d(x,y) = |x-y| \geq 0.$

$0 = d(x,y) = |x-y| \Longleftrightarrow x-y=0 \Longleftrightarrow x=y.$

(M2) $d(y,x) = |y-x| = |(-1)(x-y)| =$

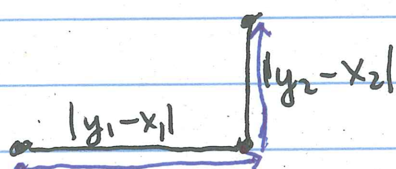
$|-1| \cdot |x-y| = |x-y| = d(x,y).$

(M3) $d(x,z) = |x-z| = |(x-y)+(y-z)| \leq$

$|x-y| + |y-z| = d(x,y) + d(y,z).$

Another metric on $\mathbb{R}^n$ comes from the taxicab norm: $|v|_1 = \sum |v_j|$ ← check this is a norm!!

in $\mathbb{R}^2$



$d_1(x,y) = |x-y|$

Geometrically $d_1(x,y)$ in $\mathbb{R}^2$ is the horizontal separation + the vertical separation.

WRITE d_2 FOR THE PYTHAGOREAN METRIC.

Some more examples Many on pp. 40-48 of Sutherland. We shall describe a few that are particularly important.

STANDARD DISCRETE METRICS X = any set

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

(Sutherland, 5.6)
(M1) & (M2) OBVIOUS

Check the Triangle $\leq$: $d(x,z) \leq d(x,y) + d(y,z)$

Case 1 $x = z \Rightarrow d(x,z) = 0$, and we know $d(x,y), d(y,z) \geq 0$, so $d(x,y) + d(y,z) \geq 0$.

Case 2 $x \neq z$ (2A) If $x \neq y$ then
$1 = d(x,z)$ and $d(x,y) + d(y,z) \geq d(x,y) = 1$,
(2B) If $y \neq z$, LHS $\geq d(y,z) = 1$ $= d(x,z)$

IMHO "pathological" (Sutherland, ¶ on pp. 41-42) is too strong, and these spaces DO arise "in nature." However, the rest of the ¶ contains some important points. — It's too soon to try and construct something which is REALLY bizarre (but it definitely can be done!).

"non-standard" might be better

METRIC SUBSPACES

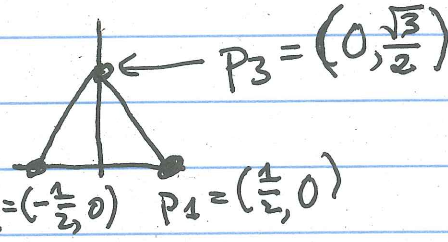
(X, d) metric space, $A \subseteq X \Rightarrow$ the subspace metric (d^A or just d) is $d^A = d^X | A \times A$. (Sutherland, 5.8)

Trivia (?) Every metric space "looks like" a subspace of a normed vector space.

<u>What does "looks like" mean?</u>

<u>ISOMETRY</u> = 1-1 onto map $f: X \to Y$ (where (X, d^X) and (Y, d^Y) are metric spaces) such that $d^Y(f(x_1), f(x_2)) = d^X(x_1, x_2)$.

For example, $\{1, 2, 3\}$ with the discrete metric is isometric to the vertices of the following equilateral triangle in $\mathbb{R}^2$:



<u>Reference for the trivia statement</u>: See the Proposition on pp. 50-51 of grad-level-classnotes.pdf (the result is useful in a few situations, but definitely not in many others)

<u>Change of scale metrics</u> If (X, d^X) is a metric space and $k > 0$, so is $(X, k \cdot d^X)$ by Exercise 5.12 in Sutherland.

<u>FUNCTION SPACES.</u> (Modified from Sutherland, 5.13) — These reflect the usefulness of metric spaces in a wide range of contexts — see 5.11 + 5.12 for others)

X = continuous real valued functions on $[0,1]$; if $f : [0,1] \to \mathbb{R}$ is continuous let

$\|f\| = \max_{t \in [0,1]} |f(x)|$, which exists since $|f|$ attains a maximum value.

(f continuous $\Rightarrow$ $|f|$ continuous). Then $\| \cdots \|$ defines a norm on X, so it makes X into a metric space.

(Sutherland, 5.10)

PRODUCT METRICS Given (X, d^X) and (Y, d^Y) define metrics on $X \times Y$ by $\boxed{z = (x,y)}$

$$d_1(z_1, z_2) = d^X(x_1, x_2) + d^Y(y_1, y_2) \quad \begin{pmatrix}\text{Taxicab} \\ \text{metric}\end{pmatrix}$$

$$d_2(z_1, z_2) = \sqrt{d^X(x_1, x_2)^2 + d^Y(y_1, y_2)^2} \quad \begin{pmatrix}\text{Pythagorean} \\ \text{metric}\end{pmatrix}$$

$$d_\infty(z_1, z_2) = \max\{d^X(x_1, x_2), d^Y(y_1, y_2)\} \quad \begin{pmatrix}\text{max} \\ \text{metric}\end{pmatrix}$$

Verifications of metric properties are in the exercises from Chapter 5 of Sutherland.

See product-metrics.pdf, product-metrics$\begin{bmatrix}2\\3\end{bmatrix}$.pdf for a family of metrics d_p, where $1 \le p \le \infty$.

Limits and continuity on metric spaces

See Def. 5.3 (p. 40) for continuity. As for limits:

<u>Def.</u> Suppose $a \in X$, metric space, and f is a function from $\{x \in X \mid 0 < d(x,a) < h\}$ to Y, another metric space. Then $\lim_{x \to a} f(x) = L$ means for each $\varepsilon > 0$ there is some $\delta > 0$ such that
$$0 < d^X(x,a) < \delta \Rightarrow d^Y(f(x), L) < \varepsilon.$$

<u>Expanded Sutherland, p. 48</u> All the results on continuous functions, and the constructions on real valued continuous fcns., carry over to metric spaces.

<u>Example not in Sutherland</u> $f : (X, d^X) \to (Y, d^Y)$ continuous at $a \in X$, and $f(a) \neq b \in Y$. Then there is some $\delta > 0$ s.t. $d(x,a) < \delta \Rightarrow f(x) \neq b$.

<u>PROOF.</u> Let $c = d^Y(f(a), b)$, and choose $\delta > 0$ so that $d(x,a) < \delta \Rightarrow d(f(x), f(a)) < \frac{c}{2}$. Then $d(x,a) < \delta \Rightarrow f(x) \neq b$, for if this were true then $d(f(x), a) = d(b, f(a)) = c$. ∎

Continuity and product constructions

Given: (X, d^X), (Y, d^Y), $z \in X \times Y$

$z = (x, y)$. d_p metric, $p = 1, 2, \infty$.

THIS TREATMENT DIFFERS FROM SUTHERLAND'S

Coordinate projections $\pi_X : X \times Y \to X$

$\pi_Y : X \times Y \to Y$

$\pi_X(x, y) = x$, $\pi_Y(x,y) = y$

Prop. 5.20 $\pi_X : (X \times Y, d_p) \to (X, d^X)$ and $\pi_Y : (X \times Y, d_p) \to (Y, d^Y)$ are continuous.

New Proof. For each choice of p we have

$d^X(x_1, x_2) \leq d_p^*(z_1, z_2)$, $d^Y(y_1, y_2) \leq d_p(z_1, z_2)$

so for each $z \in X \times Y$ we can take $\delta = \varepsilon$. ∎

Note These functions are uniformly continuous:

$g : (U, d^U) \to (V, d^V)$ is unif. cont. $\Leftrightarrow$ for each $\varepsilon > 0$ there is some $\delta > 0$ such that, for all u_1 & u_2, $d^U(u_1, u_2) < \delta \Rightarrow d^V(g(u_1), g(u_2)) < \varepsilon$.

(Usually the δ for an ε depends on x — think about $f(x) = \frac{1}{x}$ for $0 < x \in \mathbb{R}$).

THEOREM. (W, d^W), (X, d^X), (Y, d^Y) metric spaces, $f:(W,d^W) \to (X, d^X)$ and $g:(W,d^W) \to (Y,d^Y)$ cont. Define $h: W \to X \times Y$ by $h(w) = (f(w), g(w))$. Then $h:(W, d^W) \to (X\times Y, d_p)$ is continuous, where $p = 1, 2, \infty$. [Prop. 5.22 is a special case]*

Proof. Let $w \in W$ and $\varepsilon > 0$. Also, let $K(1) = \frac{1}{2}$, $K(2) = \frac{\sqrt{2}}{2}$, $K(\infty) = 1$; in each case, ~~we have~~ for $z_1 + z_2 \in X \times Y$ we have

$$d^X(x_1,x_2),\ d^Y(y_1,y_2) < \frac{\varepsilon}{K(p)} \Rightarrow d^p(z_1,z_2) < \varepsilon$$

(check this out!). By continuity there exist $\delta_X, \delta_Y > 0$ such that

$$d(t,w) < \delta_X \Rightarrow d(f(t), f(w)) < \frac{\varepsilon}{K(p)}$$

$$d(t,w) < \delta_Y \Rightarrow d(g(t), g(w)) < \frac{\varepsilon}{K(p)}$$

Let $\delta = \min\{\delta_X, \delta_Y\}$. Then by the preceding, $d(t,w) < \delta \Rightarrow \left\{ \begin{array}{l} d(f(t), f(w)) < \frac{\varepsilon}{K(p)} \\ d(g(t), g(w)) < \frac{\varepsilon}{K(p)} \end{array} \right\}$ so that $d_p(h(t), h(w)) < \varepsilon$. ∎

* Note. The diagonal $\Delta: X \to X\times X$ is obtained if $f = g =$ idenity map of X.

<u>Sutherland, Prop. 5.19</u> Let $f:(X,d^X)\to(U,d^U)$ and $g:(Y,d^Y)\longrightarrow(V,d^V)$ be continuous.
Then $f\times g\,(x,y)=(f(x),g(y))$ is continuous.

<u>New proof</u> It helps to draw a picture (a <u>commutative diagram</u>) with all the mappings, such that two composites from the same source to the same target are equal:

$$\begin{tikzcd} X \arrow[r, "f"] & U \\ X\times Y \arrow[u, "\pi_X"] \arrow[r, "f\times g"] \arrow[d, "\pi_Y"'] & U\times V \arrow[u, "\pi_U"'] \arrow[d, "\pi_V"] \\ Y \arrow[r, "g"] & V \end{tikzcd}$$

Check that the composites are equal!

In the preceding result's setting, let
$F: X\times Y\to U$ be $f\circ\pi_X$,
$G: X\times Y\to V$ be $g\circ\pi_Y$.
Then $H=f\times g$ by construction.
Now F and G are composites of cont. maps and hence are continuous. By the theorem, this means that $H=f\times g$ is also continuous. ∎

As noted in the ¶ on pp. 49–50 of Sutherland, these results yield slightly simpler (and more generalizable) proofs of statements in Prop. 5.17.

Subsets of metric spaces with special properties

The need to specify various types of subsets in (X, d^X) will arise repeatedly throughout this course (and subsequent ones). We start with a simple example generalizing a property of subsets of $\mathbb{R}$.

$A \subseteq X$ is bounded if for some $x_0 \in X$ we have $d(x_0, a) \leq K$, some $K \geq 0$ (if K works and $K' > K$ then K' also works). Say K is a bound.

Prop. 5.23A If A is bounded, then there is a constant M such that $d(a_1, a_2) \leq M$, all $a_1, a_2 \in A$.

Proof $d(a_1, a_2) \leq d(a_1, x_0) + d(x_0, a_2) \leq 2K$. ∎

The diameter of a bounded set A is the least upper bound of the set of distances $d(a_1, a_2)$ if $A \neq \emptyset$; also, $\text{diam}\, \emptyset = 0$.

Prop. 5.20 A union of two bounded sets is bounded ($\Rightarrow$ Same for finitely many, by induction).

Proof. Say $d(x_i, a_i) \leq K_i$ if $a_i \in A_i$, where $i = 1$ or 2. Then $d(x_1, a_2) \leq$

$d(x_1, x_2) + d(x_2, a_2) \leq d(x_1, x_2) + K_2$, so

$a \in A_1 \cup A_2 \Rightarrow d(x_1, a) \leq$ larger of K_1 and

$d(x_1, x_2) + K_2$. ∎

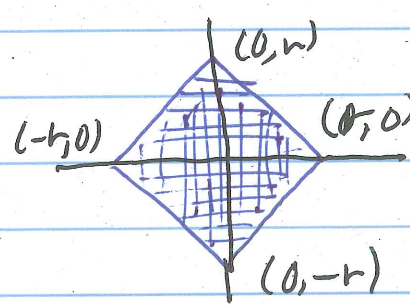
Open neighborhoods of radius r

$$N_r(x) = \{y \in X \mid d(x,y) < r\}.$$

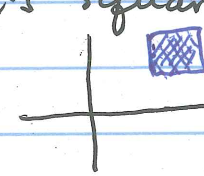
In $\mathbb{R}$ these are open intervals centered at x

In $\mathbb{R}^2$ these are open disks centered at x (with the Pythagorean metric)

In $\mathbb{R}^2$ with the taxicab metric, these are squares whose edges make $45°$ angles with the coordinate axes:



In $\mathbb{R}^2$ with the d_∞ metric, one gets squares whose edges are parallel to the axes



For discrete metrics,

$$N_r(x) = \{x\} \text{ if } r \le 1, \; X \text{ if } r > 1$$

strictly greater than

For $\mathcal{C}([0,1])$ = continuous $\mathbb{R}$ valued functions,

$N_r(f)$ = all g such that $|g(x) - f(x)| < r$ all x.

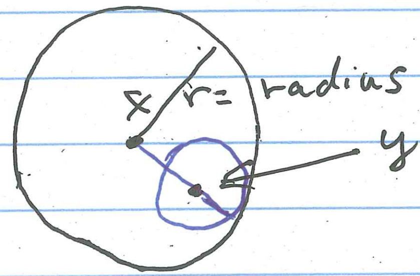
Open Sets

In $\mathbb{R}^n$, these are the sets needed to study partial derivatives.

Def. (X, d^X) metric space. $U \subseteq X$ is open in X (an open subset of X) $\Longleftrightarrow$ for each $u \in U$ there is some $\varepsilon > 0$ (depending on u) so that $N_\varepsilon(x) \subseteq U$.

Sutherland, Prop. 5.31 An open neighborhood of a point $x \in X$ is an open subset of X.

Proof. Figure 5.4 on p. 53 of Sutherland is helpful.



It suggests we take $\varepsilon = r - d(x,y)$ if $y \in N_r(x)$.

condition for $z \in N_\varepsilon(y)$

We want $N_\varepsilon(y) \subseteq N_r(x)$. But $d(z,y) < \varepsilon \Rightarrow$

$$d(z,x) \le d(z,y) + d(y,x) < (r - d(x,y)) + d(x,y) = \varepsilon. \; \blacksquare$$

(so $z \in N_r(x)$)

<u>Some open subsets of $\mathbb{R}$</u>

(a,b) — If $x \in (a,b)$ then

$N_r(x) = (x-r, x+r)$, which is contained

in (a,b) if $r \leq x-a$, ~~b-y~~ $b-x$.

Similarly for $(-\infty, b)$ (need $r \leq$ ~~b-y~~ $b-x$)

(a, ∞) (need $r \leq x-a$).

<u>A subset which is not open in $\mathbb{R}$</u>

$A = [0,1]$ is not open.

If $x = 0$, <u>every</u> set $N_r(x) = (-r, r)$

contains points not in A; specific examples

are $-\frac{r}{2}, -\frac{r}{3}, \ldots$.

More generally the $\left\{ \begin{matrix} \text{closed} \\ \text{half open} \end{matrix} \right\}$ intervals

$\left\{ \begin{matrix} [a,b] \\ [a,b) \\ (a,b] \end{matrix} \right\}$ are <u>NEVER</u> open in $\mathbb{R}$.

Basic properties of the family $\mathcal{U}_X$ of open subsets in X.

(1) $\emptyset$ and X are open

Nothing to prove $N_1(x) \subseteq X$ always

(2) A union (possibly infinite!) of open sets is open.

Say U_α open, $\alpha \in \Lambda$ $x \in \cup U_\alpha \Rightarrow x \in U_{\alpha_0}$ some α_0 and hence $N_r(x) \subseteq U_{\alpha_0}$ for some r.
But $U_{\alpha_0} \subseteq \cup U_\alpha \Rightarrow N_r(x) \subseteq \cup U_\alpha$ too. ∎

(3) An intersection of two ($\Rightarrow$ finitely many) open sets is open. BY INDUCTION

Say $x \in U_1 \cup U_2$. Choose $s, t > 0$ so $N_s(x) \subseteq U_1$ & $N_t(x) \subseteq U_2$. If $r = \min\{s, t\}$, then $N_r(x) \subseteq U_1 \cap U_2$. ∎

Corollary In a discrete metric, every subset is open.

Proof. $\{x\} = N_{\frac{1}{2}}(x) \Rightarrow \{x\}$ open all $x \in X$.

But $A \subseteq X \Rightarrow A = \bigcup_{x \in A} \{a\}$, so A is open. ∎

Continuity and open subsets

FUNDAMENTAL RESULT $f : (X, d^X) \to (Y, d^Y)$ is continuous $\Longleftrightarrow$ for ~~all~~ each U open in Y, the inverse image $f^{-1}[U]$ is open in X.

($\Rightarrow$) Suppose f is continuous.

Let $x \in f^{-1}[U]$, so that $f(x) \in U$.

Take $\varepsilon > 0$ so $N^Y_\varepsilon(f(x)) \subseteq U$. By continuity there is some $\delta_x > 0$ so that if $v \in N^X_{\delta_x}(x)$ then $f(v) \subseteq N^Y_\varepsilon(f(x))$, and thus we have

$$N^X_{\delta_x}(x) \subseteq f^{-1}[N^Y_\varepsilon(f(x))] \subseteq f^{-1}[U].$$

Hence $f^{-1}[U]$ is open in X.

($\Leftarrow$) Suppose that inverse images of open subsets are open.

x is arbitrary

Let $y \in U$ with $y = f(x)$ and $U = N^Y_\varepsilon(y)$.

Since $f^{-1}[U]$ is open, there is some $\delta > 0$ so that $N_\delta(x) \subseteq f^{-1}[U]$. Then $d(v, x) < \delta \Rightarrow v \in N_\delta(x) \Rightarrow f(v) \in U = N^Y_\varepsilon(f(x))$, so that $d(f(v), f(x)) < \varepsilon$. Hence f is continuous at $x \in X$. ∎

Finally, some warnings.

1. An infinite intersection of open sets is not necessarily open. Take $U_n \subseteq \mathbb{R}$ with $U_n = (-\frac{1}{n}, \frac{1}{n})$, so $\bigcap_n U_n = \{0\}$. The latter does not contain any subsets of the form $N_r(0) = (-r, r)$ for $r > 0$ because $\frac{r}{2} \notin \{0\}$.

2. As the course progresses we often suppress d^X from (X, d^X) and simply talk about X as a metric space. However, properties of a subset like openness or boundedness depend heavily on the underlying metric. See Sutherland, Example 5.36.

3. If $f : (X, d^X) \to (Y, d^Y)$ is continuous, then U open in X does not imply $f[U]$ is open in Y and vice versa.

   A. $X = Y = \mathbb{R}$, $f =$ constant map w/ value $\{0\}$. Then U open nonempty $\Rightarrow f[U] = \{0\}$, which we have seen is not an open subset of $\mathbb{R}$.

   B. Here is a map f such that U open in $X \Rightarrow f[U]$ is open in Y, but f is not continuous:

5.17

Let $X = (\mathbb{R}, d^X)$ $\quad d^X$ = usual metric
$\quad\;\; Y = (\mathbb{R}, d^Y)$ $\quad d^Y$ = discrete metric

~~[illegible]~~ $f : X \longrightarrow Y \qquad f(x) = x.$

Then if U is d^X open, $U = f[U]$ is also d^Y open because <u>all</u> subsets are d^Y open. However, $\{0\}$ is d^Y open but not d^X open, and hence f is not continuous.

<u>Definition</u> If U open in $X \Longrightarrow f[U]$ open in Y, we say that f is an <u>open mapping</u>.

<u>Examples</u> $\pi_X : (X \times Y, d_\infty) \longrightarrow X$
$\qquad\qquad\;\; \pi_Y : (X \times Y, d_\infty) \longrightarrow Y$ $\quad$ are open.

We only check the first; the second is similar.

$z = (x, y) \in X \times Y$, $\quad N^\infty_\varepsilon(z) = d_\infty$ neighborhood.

$\pi_X [N^\infty_\varepsilon(z)] = N_\varepsilon(x)$ because
$$N^\infty_\varepsilon(z) = N_\varepsilon(x) \times N_\varepsilon(y).$$

If $U \subseteq X \times Y$ is open and $N^\infty_{\delta(z)}(z) \subseteq U$, then
$$U = \bigcup_z N^\infty_{\delta(z)}(z) \text{ and hence}$$

See the Additional Exercises

$$\pi_X[U] = \pi_X\left[\cup N^{\infty}_{\delta(z)}(z)\right] =$$

$$\pi_X\left[\cup N_{\delta(z)}(\pi_X(z)) \times N_{\delta(z)}(\pi_Y(z))\right] =$$

$$\bigcup_z N_{\delta(z)}(\pi_X(z)),$$ which is a union of open sets and hence is open in X. ∎

Examples with no proof.

$f : \mathbb{C} \longrightarrow \mathbb{C}$ complex polynomial function (non constant)

(shown in courses on complex variables)