6. More Concepts in Metric Spaces

Slightly out of order from Sutherland.

(1) Infinite sequences in metric spaces
(2) Closed regions and subsets
(3) Associated closed and open subsets
(4) Reversible change of variables transformations

The definition of limit for a sequence $\{a_n\}$ in a metric space (X,d) is the same as for $\mathbb{R}$.

$\lim_{n\to\infty} a_n = \ell \Leftrightarrow$ for each $\varepsilon > 0$ there is some $N \in \mathbb{N}^+$ such that $k \geq N \Rightarrow d(a_k, \ell) < \varepsilon$.

Just like proofs in $\mathbb{R}$.

<u>Prop. 6.26</u> There is at most one limit value.

<u>Ex. 6.25</u> $f: (X, d^X) \to (Y, d^Y)$ is continuous $\Leftrightarrow$ for all sequences, $\lim_{n\to\infty} a_n = a \Rightarrow \lim_{n\to\infty} f(a_n) = f(a)$.

<u>Note</u>. We shall not need Cauchy sequences in this course. These are sequence with the the following property:

For each $\varepsilon > 0$ there is some M such that $n, m \geq M \Rightarrow d(a_n, a_m) < \varepsilon$.

Ex. 6.24 A convergent sequence is Cauchy

The converse is false. Take $X = \mathbb{R} - \{0\}$ with the usual metric, $a_n = \frac{1}{n}$. Then $\lim_{n \to \infty} a_n$ does not exist in X. However,

if (X, d) is a metric space, then X is isometric to a subset of a metric space (X^*, d^*) in which every Cauchy sequence converges (see the Prop. on pp. 50-51 of grad-level-classnotes.pdf).

(2) CLOSED SUBSETS.

Standard domains for double integration have the form $a \leq x \leq b$, $g(x) \leq y \leq f(x)$ where g & f are continuous. — These are the main examples.

Definition (X, d) metric space, then $A \subseteq X$ is closed $\Leftrightarrow$ $X - A$ is open.

Examples in $\mathbb{R}$

$[a, b]$, for $\mathbb{R} - [a, b] = (-\infty, a) \cup (b, \infty)$

$[a, \infty)$

$(-\infty, b]$

$A = [0,1] \cup [2,3]$, for $\mathbb{R} - A = (-\infty, 0) \cup (1,2) \cup (3, \infty)$

(Recall that open intervals are open in $\mathbb{R}$)

Note that a subset can be neither open nor closed. For example, $A = [0,1) \subseteq \mathbb{R}$.

Then $X - A = (-\infty, 0) \cup [1, \infty)$

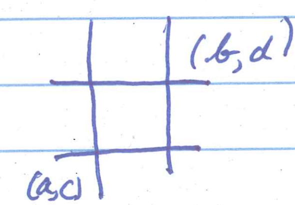
No set $N_r(1)$ is contained in either A or $X-A$.

More examples.

6.2. (a). (iv) $X = \mathbb{R}$, $A = \{\frac{1}{n} \mid n \in \mathbb{N}^+\} \cup \{0\}$,

for $X - A = (-\infty, 0) \cup (1, \infty) \cup \left(\bigcup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n})\right)$

(c) $X = \mathbb{R}^2$, $A = [a,b] \times [c,d]$, for



$X - A = (-\infty, a) \times \mathbb{R} \cup$
$(b, \infty) \times \mathbb{R} \cup$
$\mathbb{R} \times (-\infty, c) \cup$
$\mathbb{R} \times (d, \infty)$

and U open in X, V open in $Y \Rightarrow$

$U \times V = \pi_X^{-1}[U] \cap \pi_Y^{-1}[V]$ is open in the d_p metric if $p = 1, 2, \infty$.

6.2. (d) If X has the discrete metric, then every subset is closed (since every subset is open).

<u>Variation on 6.2.(e)</u> If $f: X \to \mathbb{R}$ is continuous, then $f^{-1}[\{0\}]$ is closed, for $X - f^{-1}[\{0\}] = f^{-1}[\mathbb{R}-\{0\}]$ and $\mathbb{R}-\{0\} = (-\infty,0)\cup(0,\infty)$ is open.

Similarly for $f^{-1}[\,[0,\infty)]$, $f^{-1}[(-\infty,0]]$
[Complements are $(-\infty,0)$ & $(0,\infty)$]

More generally,

<u>Prop. 6.6</u> $f: X \to Y$ is continuous $\Leftrightarrow$ for all closed $B \subseteq Y$, $f^{-1}[B]$ is closed in X.

<u>Idea of proof</u> Use $f^{-1}[Y-B] = X - f^{-1}[B]$. ∎

★ We want to show that if $\begin{matrix} f: \\ g: \end{matrix} [a,b] \to \mathbb{R}$ ~~are~~ continuous, then $\{(x,y) \mid a \le x \le b,\ g(x) \le y \le f(x)\}$ is ~~are~~ closed in $\mathbb{R}^2$. ~~When does a point lie in its complement?~~.

<u>Preliminary steps</u> Additional exercise 5.7 says f extends to all of $\mathbb{R}$ continuously.

In $\mathbb{R}^2$ we have
$$y \ge g(x) \Leftrightarrow y - g(x) \ge 0$$
$$y \le f(x) \Leftrightarrow f(x) - y \ge 0$$

so $\begin{matrix} \{(x,y) \mid y-g(x)\} \\ \{(x,y) \mid f(x)-y\} \end{matrix}$ are closed since $\left.\begin{matrix} y-g(x) \\ f(x)-y \end{matrix}\right\}$ cont.

- Likewise, $\{(x,y) \mid x \le b\}$ and $\{(x,y) \mid a \le x\}$ are closed.

<u>Prop. 6.</u> (a) ϕ, X are closed in X

(b) An arbitrary intersection of closed sets is closed.

(c) A finite (or 2 fold) union of closed sets is closed.

<u>Proof.</u> (a) $X - X = \phi$, $X - \phi = X$ and ϕ, X are open in X.

(b) $\{F_\alpha\}$ closed $\Rightarrow X - \cap F_\alpha = \cup_\alpha X - F_\alpha =$ union of open sets.

(c) $F_1 + F_2$ closed $\Rightarrow X - (F_1 \cup F_2) = (X - F_1) \cap (X - F_2) =$ ~~union of~~ intersection of two open sets. ∎

Hence $\{(x,y) \mid a \le x \le b, g(x) \le y \le f(x)\}$ is the intersection of $\{\ldots \mid a \le x\}$, $\{\ldots \mid x \le b\}$, $\{g(x) \le y\}$ and $\{\ldots \mid y \le f(x)\}$ and each of these subsets is closed. ∎

The origin of the named "closed" comes from the fact that $A \subseteq X$ is closed $\Leftrightarrow$ A is "closed under taking limits of convergent sequences in X."

Sutherland, final sentence in Exercise 6.26

$A \subseteq X$ is closed $\Longleftrightarrow$ for each sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n = b \in X$, the limit is in A.

Proof.

($\Longrightarrow$) Suppose $\lim_{n\to\infty} a_n = b \in X - A$, we are assuming that A is a closed subset. Choose $\varepsilon > 0$ so that $N_\varepsilon(b) \subseteq X - A$. By the definition of limit we have $a_n \in N_\varepsilon(b) \subseteq X - A$ for $n \geq N$. But this contradicts $a_n \in A$ for all n. Hence $\lim_{n\to\infty} a_n \in A$. ∎

($\Longleftarrow$) Suppose A is not closed, so $X - A$ is not open. Then there is <u>some</u> $b \in X - A$ such that no set of the form $N_\varepsilon(b)$ is contained in $X - A$.

In other words, for each $\varepsilon > 0$ there is a point x_ε in $A \cap N_\varepsilon(b)$. Construct a sequence recursively: PICK $x_1 \in A \cap N_1(b)$. Given $x_n \in A$ s.t. $d(x_n, b) < \frac{1}{n}$, choose $x_{n+1} \in N_\varepsilon(b)$ where $\varepsilon = \min\{\frac{1}{2}d(x_n, b), \frac{1}{n+1}\}$.

Then $d(x_n, b) < \frac{1}{n}$ for all n, so $x_n \in A$ and $\lim_{n\to\infty} x_n = b \notin A$. ∎

(3) <u>Note</u> If $A \subseteq \mathbb{R}^2$ is the closed region $a \le x \le b$, $g(x) \le y \le f(x)$, AND $g(x) < f(x)$ for $x \in (a,b)$, then $\{(x,y) \mid a < x < b,\ g(x) < y < f(x)\}$ is an <u>open</u> subset of $\mathbb{R}^2$.

<u>Proof.</u>
$$\left.\begin{array}{l} \{(x,y) \mid y > g(x)\} = \{\ldots \mid y - g(x) > 0\} \\ \{(x,y) \mid f(x) > y\} = \{\ldots \mid f(x) - y > 0\} \\ \{(x,y) \mid a < x < b\} \end{array}\right] \text{are all open sets}$$

(1st. two are inverse images of $(0,\infty)$ for cont. fcns. on $\mathbb{R}^2$), and the intersection of finitely many open sets is open.

<u>Suggests</u> In many cases, given a subset in $\mathbb{R}^2$ we have a naturally associated $\left\{\begin{array}{c}\text{open}\\\text{closed}\end{array}\right\}$ subset.

$\boxed{\text{This is what we shall analyze next.}}$

<u>RELATED CONCEPT</u> <u>boundary</u> or <u>frontier</u> between $A \subseteq X$ and $X - A \subseteq X$.

$A \subseteq X$

Closure of $A = \bar{A}$, smallest closed set cont. $A =$

$$\bigcap_{\substack{F \text{ closed} \\ F \supseteq A}} F \quad \left(\begin{array}{c}\text{so there is a} \\ \text{unique smallest} \\ \text{set}\end{array}\right)$$

Interior of $A = \mathring{A} = \mathrm{Int} A$, largest open subset contained in $A =$ $\bigcup_{\substack{V \text{ open} \\ V \subseteq A}} U$ $\left(\begin{array}{c}\Rightarrow \text{a unique} \\ \text{largest set}\end{array}\right)$.

With hindsight, it's easier to take an indirect approach and start by looking at another concept.

Definition 6.15 (X, d) metric space, $A \subseteq X$. Then $p \in X$ is a limit point of A if for each $\varepsilon > 0$, $(N_\varepsilon(p) - \{p\}) \cap A \neq \emptyset$. Notice that a point in A is not necessarily a limit point! $X = \mathbb{R}$, $A = \{0\}$ no limit points.

limit point = accumulation point

Example Limit points of (a, b) are $[a, b]$.

Explanation and verification.

First notice that in the definition, we can replace ε with all sufficiently small ε because $\varepsilon < \varepsilon' \Rightarrow N_\varepsilon \subseteq N_{\varepsilon'}$.

If $x \in (a,b)$ and $(x-\delta, x+\delta) \subseteq (a,b)$, then $(x-\delta, x+\delta)$ contains many points of (a,b) besides x itself.

If $x = a$ or b, and $\delta < b-a$, then $N_\delta(x)$ also contains infinitely many points of (a,b).

If $x < a$ or $x > b$, then $x \in (-\infty, a) \cup (b, \infty)$, which is open. Hence there is some $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq (-\infty, a) \cup (b,\infty)$ and hence $N_\varepsilon(x) \cap [a,b] = \phi$, so x is not a limit point.

$L(A)$ = set of limit points (= <u>derived set</u>).

<u>Proposition (Counterpart to Sutherland, 6.14)</u>

$A \subseteq X$ metric $\Rightarrow$

(i) $L(A)$ is closed in X.

(ii) A is closed in $X \Leftrightarrow L(A) \subseteq A$.

(iii) $A \cup L(A)$ is closed in X.

Lemma $x \in L(A) \iff$ there is some sequence $\{a_n\}$ in A such that $a_n \neq x$ all n and $x = \lim_{n\to\infty} a_n$.

Proof of Lemma ($\Rightarrow$) Take a_1 in $(N_1(x) - \{x\}) \cap A$, which is nonempty. Suppose we have a_k for $k \leq n$ with $d(x, a_k) < \frac{1}{k}$. Let $\varepsilon = \min\{d(x,a_k), \frac{1}{k+1}\}$. Pick $a_{k+1} \in (N_\varepsilon(x) - \{x\}) \cap A$. Then (as in proof on p. 6.6 of the notes) $a_n \to a$. ∎

($\Leftarrow$) Given $\varepsilon > 0$, choose N such that $n \geq N \Rightarrow d(x, a_n) < \varepsilon$. Then $a_n \in (N_\varepsilon(x) - \{x\}) \cap A$. ∎

Cor. If $x \in L(A)$, then every $N_\varepsilon(x) - \{x\}$ contains infinitely many points of A.

(Look at the a_n in the sequence!!).

Note also that $B \subseteq A \Rightarrow L(B) \subseteq L(A)$.

PROOF OF PROPOSITION. First prove (ii), then (i), then (iii).

(ii) ($\Rightarrow$) Suppose A is closed, so that $X-A$ is open. Want to show $L(A) \cap X-A = \phi$.

But $y \in X-A \Rightarrow$ some $N_\varepsilon(y) \subseteq X-A$ $\Rightarrow (N_\varepsilon(y)-\{y\}) \cap A = \phi \Rightarrow y \notin L(A)$. ∎

($\Leftarrow$) Suppose $L(A) \subseteq A$. Want to show $X-A$ is open in X. But then $y \in X-A \Rightarrow$ $y \notin L(A) \Rightarrow$ for some $\varepsilon > 0$, $(N_\varepsilon(y)-\{y\}) \cap A$ $= \phi$, so that also $N_\varepsilon(y) \cap A = \phi$ or $N_\varepsilon(y) \subseteq X-A$. ∎

(i) By (ii), we need only show that $L(L(A)) \subseteq L(A)$. Suppose we have $b_n \to c$ where $b_n \neq c$ and $b_n \in L(A)$. Then we can find $a_n \neq b_n$ s.t. $a_n \in A$ and $d(a_n, b_n) < \begin{cases} \frac{1}{n} \\ d(b, b_n) \end{cases}$. Claim $a_n \neq b$

$a_n \to \cancel{b}$.

$a_n \neq b$ because $d(a_n, b_n) < d(b, b_n)$

To verify $a_n \to b$, let $\varepsilon > 0$.

$\varepsilon > 0 \Rightarrow \frac{\varepsilon}{2} > 0$ "trick"

Then $d(b_n, b) < \frac{\varepsilon}{2}$ if $n \geq M_1$

$d(a_n, b_n) < \frac{\varepsilon}{2}$ if $n \geq M_2$ $\Rightarrow$

if $n \geq \max\{M_1, M_2\}$ then $d(a_n, b) \leq$

$d(a_n, b_n) + d(b_n, b) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. ■

(iii) Show the complement is open.

Let $y \notin A \cup L(A)$ but $y \in X$. Since $y \notin L(A)$, for some $\varepsilon > 0$ $(N_\varepsilon(y) - \{y\}) \cap A = \emptyset$, and in fact $N_\varepsilon(y) \cap A = \emptyset$ because $y \notin A$. Hence $X - A$ is open in X. ■

Replace with 12A ↓

Corollary $\bar{A} = A \cup L(A)$.

CLOSURE OF $A = \bar{A}$, $Cl(A)$

Proof $\bar{A} \subseteq A \cup L(A)$ since RHS is closed and contains A. Conversely, if F closed & $A \subseteq F$, then $L(A) \subseteq L(F) \subseteq F$, so $A \cup L(A) \subseteq F$. ■

Why F for closed set? French for closed is fermé

Strictly speaking, we want $Cl(A; X)$, closure of A in X.

CLAIM: $N_\varepsilon(x) \cap L(A) = \phi$ too.

If so, then $N_\varepsilon(x) \cap (A \cup L(A)) = \phi$, which implies that $X - (A \cup L(A))$ is open.

But suppose $y \in N_\varepsilon(x) \cap L(A)$ and take $r > 0$ so that $N_r(y) \subseteq N_\varepsilon(x)$. Then $(N_r(y) - \{y\}) \cap A \neq \phi$, so that $N_\varepsilon(x) \cap A \neq \phi$, contradiction. The source of the contradiction was the assumption that $N_\varepsilon(x) \cap L(A) \neq \phi$, so $N_\varepsilon(x) \cap A = \phi$ as claimed. ∎

Why F for a closed set? The French term is fermé.

Corollary $\bar{A} = A \cup L(A)$.

Proof. $\bar{A} \subseteq A \cup L(A)$ since the right side is closed in X and it contains A. Conversely, if $A \subseteq F$ closed, then $L(A) \subseteq L(F) \subseteq F$, so $A \cup L(A) \subseteq F$. Hence $A \cup L(A) \subseteq \bigcap_{\substack{F \supseteq A \\ \text{closed}}} F = \bar{A}$. ∎

Strictly speaking, $A \cup L(A) = Cl(A; X)$, the closure of A in X.

More on closures, etc.

A is dense in X if $\overline{A} = X$

X is *dense-in-itself* if $X = L(X)$.

Sutherland, Prop. 6.11 (partial), ~~to~~ 6.13, 6.14:

(i) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$

(ii) $\overline{\overline{A}} = \overline{A}$

(iii) $\overline{\bigcap_{\alpha} A_\alpha} \subseteq \bigcap_{\alpha} \overline{A_\alpha}$

(iv) $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$.

<u>Note</u> Containment in (iii) may be proper. Let $X = \mathbb{R}$, $A_1 = (-\infty, 0)$
$A_2 = (0, \infty)$ so

$A_1 \cap A_2 = \phi \Rightarrow A_1 \cap A_2 = \overline{A_1 \cap A_2}$, but

$0 \in \overline{A_1} \cap \overline{A_2}$.

<u>Proofs</u>. (i) $A \subseteq B \subseteq \overline{B}$, so $\overline{B}$ is a closed subset containing A, so $\overline{A} \subseteq \overline{B}$. $\blacksquare$

(ii) $\overline{\overline{A}}$ is a closed set containing $\overline{A}$, so

$\overline{A} \subseteq \overline{\overline{A}}$, and $\overline{\overline{A}} = \overline{A} \cup L(\overline{A}) \subseteq \overline{A} \cup \overline{A} = \overline{A}$. ∎

(iii) RHS is a closed subset containing the intersection, so it also contains the latter's closure. ∎

(iv) By (i), $\overline{A_i} \subseteq \overline{A_1 \cup A_2}$, so

$\overline{A_1} \cup \overline{A_2} \subseteq \overline{A_1 \cup A_2}$. But $\overline{A_1} \cup \overline{A_2}$ is closed and contains $A_1 \cup A_2$, so also

$\overline{A_1 \cup A_2} \subseteq \overline{A_1} \cup \overline{A_2}$. ∎

<u>Sutherland, Prop. 6.12</u> $f: (X, d^X) \to (Y, d^Y)$ is continuous $\iff$ for all $A \subseteq X$ we have

$f[\overline{A}] \subseteq \overline{f[A]}$. (Proved in Exercise 6.13).

<u>INTERIOR OF $A \subseteq X$</u> $\mathring{A}$, A°, or

$\mathrm{Int}(A, X)$, or sometimes $\mathrm{Int}(A)$ — [if no ambiguity about X]

Formally, it is all $a \in A$ such that some $N_\varepsilon(a) \subseteq A$.

EXAMPLES 1. The interior of $[a, b]$ is (a, b).

If $x \in (a,b)$, then some $(x-\delta, x+\delta) \subseteq (a,b) \subseteq [a,b]$, so $(a,b) \subseteq$ Interior.

On the other hand, for each $\varepsilon > 0$ the sets $N_\varepsilon(a)$, $N_\varepsilon(b)$ are not contained in $[a,b]$, so $a, b \notin \mathrm{Int}([a,b])$.

2. The interior of $\{0\}$ in $\mathbb{R}$ is empty, for $N_\varepsilon(0)$ is never cont. in $\{0\}$.

3. The interior of $\mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$ is also empty for the same reason (here, $N_\varepsilon((t,0))$ contains $(t, \varepsilon/2)$).

<u>CLAIM</u> $\mathrm{Int}(A, X)$ is the unique largest open subset $U \subseteq X$ such that $U \subseteq A$.

<u>Verification</u> (i) U open $+ U \subseteq A \Rightarrow U \subseteq \mathrm{Int}\,A$, for if $a \in U$ open $+ U \subseteq A$, then for some $\varepsilon > 0$ we have $N_{\varepsilon(a)}(a) \subseteq U \subseteq A$, and hence $a \in \mathrm{Int}\,A$. ∎

(ii) We shall prove $\mathrm{Int}(A, X)$ is open in X.

Let $a \in \mathrm{Int}(A, X)$, $N_{\varepsilon(a)}(a) \subseteq A$; we shall show $N_\varepsilon(a) \subseteq \mathrm{Int}\,A$. But $y \in N_{\varepsilon(a)}(a)$ and $\delta = d(a, y) \Rightarrow N_{\varepsilon(a)-\delta}(y) \subseteq N_{\varepsilon(a)}(a) \subseteq A$, so $y \in \mathrm{Int}\,A$. Finally, we use this to see

$$\mathrm{Int}\,A = \bigcup_{a \in \mathrm{Int}\,A} \{a\} \subseteq \bigcup_{a \in \mathrm{Int}\,A} N_{\varepsilon(a)}(a) \subseteq \mathrm{Int}\,A$$

so that $\mathrm{Int}\,A$ is the open set $\bigcup_{a \in \mathrm{Int}\,A} N_{\varepsilon(a)}(a)$. ∎

<u>More on interiors (Sutherland, 6.21)</u>

(i) $A \subseteq B \Rightarrow \mathrm{Int}(A) \subseteq \mathrm{Int}(B)$.

(ii) $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$.

Proofs

(i) $\mathrm{Int}(A) \subseteq A \subseteq B$, so $\mathrm{Int}(A)$ is an open set contained in B. Since $\mathrm{Int}(B)$ is the largest such open set, $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$. ∎

(ii) $\mathrm{Int}(A) \subseteq A$ & (i) imply $\mathrm{Int}(\mathrm{Int}(A)) \subseteq \mathrm{Int}(A)$. To finish, note that if U is open, then $\mathrm{Int}(U) = U$ follows from the definitions. ∎

Boundary or frontier points.

Warning In topology, ∂A often has another meaning, and similarly for boundary.

Def. 6.22 $A \subseteq X$ metric. The frontier or boundary of A in X, written $\mathrm{Bdy}(A;X)$ or $\mathrm{Fr}(A;X)$ is $\overline{A} - \mathrm{Int}(A)$.

Example. The frontier of (a,b) or $[a,b]$ in $\mathbb{R}$ is equal to $\{a,b\}$.

Examples

1. The frontier points of $[a, b]$ and (a, b) are $\{a, b\}$.

2. The frontier points of $\mathbb{Q}$ are all of $\mathbb{R}$, for $\overline{\mathbb{Q}} = \mathbb{R}$ ~~and~~ (since $L(\mathbb{Q}) = \mathbb{R}$!) and no set of the form $N_\varepsilon(q)$ is contained in $\mathbb{Q}$ (between q and $q+\varepsilon$ there is some irrational number).

Sutherland, Prop. 6.24 Let $x \in X$, ~~$x \in$~~ $A \subseteq X$ metric. Then $x \in \mathrm{Bdy}(A, X) \iff$ for all $\varepsilon > 0$ both $A \cap N_\varepsilon(x)$ and $(X - A) \cap N_\varepsilon(x)$ are nonempty.

Proof. $(\Rightarrow)$ $x \in \mathrm{Bdy}(A, X) \Rightarrow x \in \overline{A} - \mathrm{Int}\, A$. Since $x \in \overline{A}$, for each $\varepsilon > 0$ the set $N_\varepsilon(x) \cap A$ is nonempty (two cases: $x \in A$, $x \in L(A)$). Since $x \notin \mathrm{Int}\, A$, there is no $\varepsilon > 0$ s.t. $N_\varepsilon(x) \subseteq A$. In other words, for each $\varepsilon > 0$ there is at least one point in $N_\varepsilon(x) \cap (X - A)$. ∎

($\Leftarrow$) If $N_\varepsilon(x) \cap A$ is nonempty for all ε, then either $x \in A$ or $x \in L(A)$, so $x \in \bar{A}$.
If $N_\varepsilon(x) \cap (X - A) \neq \phi$ for all ε, then $x \notin \operatorname{Int} A$. ∎

<u>EQUIVALENT METRICS</u> Postponed until next chapter with one exception.

<u>Def.</u> (X, d^X) + (Y, d^Y) metric spaces.
$f : (X, d^X) \to (Y, d^Y)$ is a <u>homeomorphism</u>
~~if both f and its in~~
if f is 1-1 onto and both f and its inverse are continuous.

<u>THIS IS A FUNDAMENTAL CONCEPT IN TOPOLOGY!</u>

Special cases have arisen in earlier courses.

<u>Single variable calculus</u> Change of variables.

$u(x)$ continuous derivative, strictly increasing on $[a, b] \Longrightarrow u$ has a continuous derivative and a strictly increasing inverse.

One use is to rewrite integrals into more computable forms:

$$\int_a^b g(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} g(u)\,du.$$

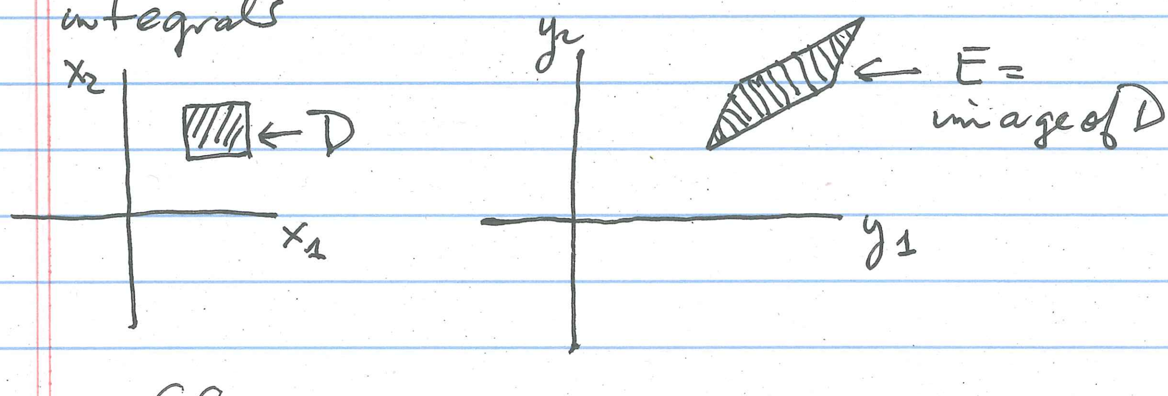
<u>Linear algebra</u> Geometric transformations of $\mathbb{R}^n$, $n=2,3,\ldots$

$$n=2 \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

where $\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{=A}$ is invertible.

$$Y = AX + B \implies \text{inverse map is}$$
$$X = A^{-1}Y - A^{-1}B.$$

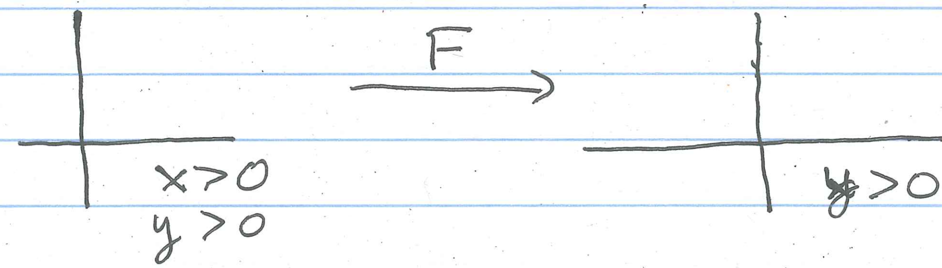
These can also be used to rewrite (double) integrals



$$\iint_E f(y_1,y_2)\,dy_1\,dy_2 = \iint_D f(y_1(x_1,x_2), y_2(x_1,x_2))\,|\det A|\,dx_1\,dx_2.$$

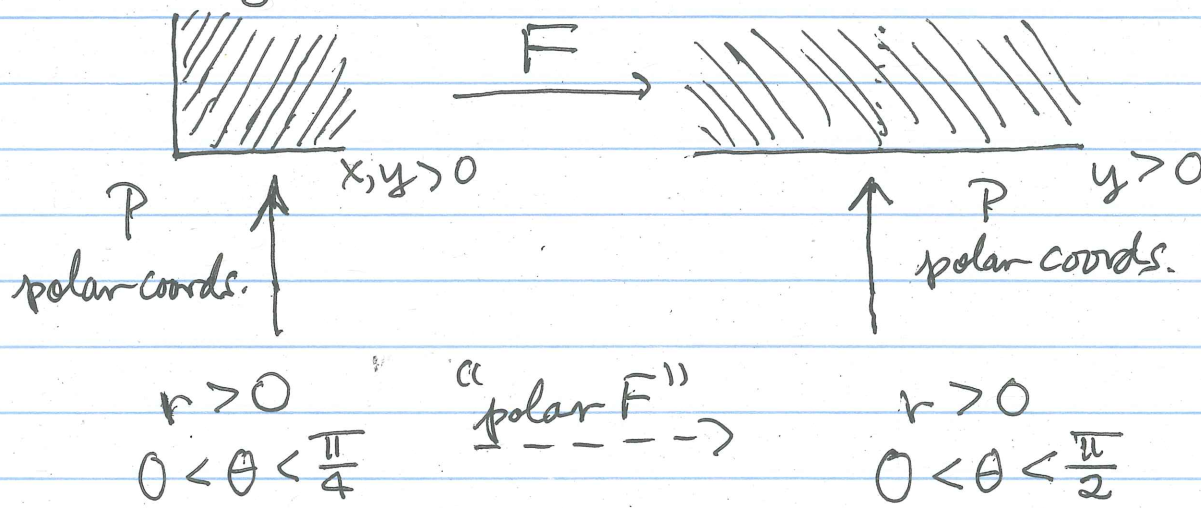
One also has more complicated 2D examples.

$(u(x,y), v(x,y)) = F(x,y) =$
$(x^2-y^2, 2xy)$ on a piece of $\mathbb{R}^2$



In polar coordinates, F sends (r,θ) to $(r^2, 2\theta)$.

Checking that F has a continuous inverse.



The maps P have a continuous inverse
$Q(x,y) = (r,\theta)$ where $r = \sqrt{x^2+y^2}$
$$\theta = \arccos \frac{x}{\sqrt{x^2+y^2}}$$

In polar coords the inverse sends (r,θ) to $(\sqrt{r}, \theta/2)$.

We can write the inverse to F more explicitly as $P(\sqrt{r}, \frac{1}{2}\theta)$ where ~~r~~ and θ are given as above in terms of x and y.

See

intro2topA-08.pdf

for more on this topic.