7. Topological Spaces

An ultimate abstract framework for continuous mappings, which is also very effective and useful.

Motivations

1. Continuity can be expressed entirely in terms of open subsets.
2. Given (X, d^X) and (Y, d^Y), the d_p metrics $(p = 1, 2, \infty)$ yield the same open sets
3. For finite metric spaces, all subsets are open.

To prove 2, use

$$\tfrac{1}{2} d_2 \le \tfrac{1}{2} d_1 \le d_\infty \le d_2 \le d_1 \le 2 d_\infty$$

plus

Sutherland, Prop. 6.34 X with metrics d + d' and $0 < m < M$ s.t. $m d' \le d \le M d'$. Then (X, d) and (X, d') have the same open sets.

Proof It suffices to prove the identity maps

$$\left\{\begin{array}{l} J_X : (X,d) \longrightarrow (X,d') \\ J'_X : (X,d') \longrightarrow (X,d) \end{array}\right\}$$ are (uniformly) continuous,

for J_X continuous $\Rightarrow$ every d'-open set is d-open

J'_X continuous $\Rightarrow$ every d-open set is d'-open.

Let $\varepsilon > 0$. Then $d'(u,v) < \frac{\varepsilon}{M} \Rightarrow d(u,v) < \varepsilon$

$d(u,v) < m\varepsilon \Rightarrow d'(u,v) < \varepsilon$. ∎

Note also $N_{\varepsilon/M}(p;d') \subseteq N_\varepsilon(p;d)$,

$N_{m\varepsilon}(p;d) \subseteq N_\varepsilon(p;d')$.

Def. A topological space is a pair $(X,\mathcal{T})$ consisting of a set X and a family of subsets $\mathcal{T}$, the $\left\{\begin{array}{l}\text{topology for} \\ \text{open subsets of}\end{array}\right\}$ X, such that

(Top 1) $\phi, X \in \mathcal{T}$.

(Top 2) $U_1, U_2 \in \mathcal{T} \Rightarrow U_1 \cap U_2 \in \mathcal{T}$.

(Top 3) $U_\alpha \in \mathcal{T}$ for $\alpha \in A \Rightarrow \bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.

In point set topology T_1, T_2, T_3 have other meanings!

Open subsets of a metric space are the obvious examples.

<u>Examples not coming from metric spaces</u>

<u>Indiscrete topology:</u> S set, take $\mathcal{T} = \{S, \phi\}$ only. If $|S| \geq 2$ this does not come from a metric space because $S - \{p\}$ is always open if (S, d) = metric space + $p \in S$.

<u>Sierpiński Space:</u> $X = \{0,1\}$, $\mathcal{T} = \phi, X, \{1\}$. Not metric, same reason.

<u>Cofinite Topology:</u> S = set, $\mathcal{T}$ = all U s.t. $X - U$ is finite <u>plus</u> X itself. If $|X| = \infty$, then this does not come from a metric space (Sutherland, Example 11.6, p. 110).

<u>Sutherland, Prop. 7.2</u> X top sp., $V \subseteq X$. Then V is open $\Leftrightarrow$ for each $x \in V$ there is an open subset U_x s.t. $x \in U_x \subseteq V$.

PROOF ($\Rightarrow$) For each $x \in V$ take $U_x = V$.

($\Leftarrow$) We have the familiar inclusion chain

$$V = \bigcup_{x \in V} \{x\} \subseteq \bigcup_{x \in V} U_x \subseteq V, \text{ so } V = \bigcup_{x \in V} U_x$$

is a union of open sets and hence is open. ∎

<u>Note</u> For the discrete metric, EVERY subset is open.

<u>Alternate description of topologies</u>

<u>Theorem</u>. Let X be a set, and let $\mathcal{F}$ be a family of subsets such that

(T1*) $\phi, X \in \mathcal{F}$

(T2*) A union of two sets in $\mathcal{F}$ is also in $\mathcal{F}$

(T3*) An arbitrary nonempty* intersection of sets in $\mathcal{F}$ is also in $\mathcal{F}$.

* it is best not to talk about the intersection of an empty family of sets!!

Then there is a unique topology $\mathcal{T}$ on X such that $U \in \mathcal{T} \Leftrightarrow X - U \in \mathcal{F}$.

<u>Proof</u> (<u>Top 1</u>) $\phi, X \in \mathcal{F} \Rightarrow$

$$X = X - \phi \text{ and } \phi = X - X \in \mathcal{T}.$$

(Top 2) If $U_i = X - F_i \in \mathcal{T}$ for $i = 1$ or 2, then $U_1 \cap U_2 = X - (F_1 \cup F_2) \in \mathcal{T}$ because $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cup F_2 \in \mathcal{F}$.

(Top 3) Suppose $U_\alpha = X - F_\alpha \in \mathcal{T}$, so $F_\alpha \in \mathcal{F}$ for all α. Then $\bigcup_\alpha U_\alpha = \bigcup_\alpha X - F_\alpha =$

$X - (\bigcap_\alpha F_\alpha)$. Since $\bigcap_\alpha F_\alpha \in \mathcal{F}$, we have $\bigcup U_\alpha \in \mathcal{T}$. ∎

ZARISKI TOPOLOGY ON $\mathbb{F}^n$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$S =$ subset of the polynomial ring $\mathbb{F}[t_1, \ldots, t_n]$. Let $V(S) =$ variety of S, $=$ all $a = (a_1, \ldots, a_n) \in \mathbb{F}^n$ such that $h(a) = 0$ for all $h \in S$ (also called the zero set of S).

Claim The $V(S)$ are the closed subsets of a topology on $\mathbb{F}^n$.

(Top 1*) $\mathbb{F}^n = V(\phi)$, $\phi = V(\text{all polys.})$

(Top 2*) Claim $V(S_1 \cdot S_2) = V(S_1) \cup V(S_2)$, where $S_1 \cdot S_2$ = all polys $g_1 \cdot g_2$ where $g_i \in S_i$ $(i = 1, 2)$.

<u>Proof</u> Check that $V(S_i) \subseteq V(S_1 \cdot S_2)$ for $i = 1$ or 2 because $g_i(a) = 0 \Rightarrow$ $i = 1$ or 2
$g_1(a)\, g_2(a) = 0$. Therefore $V(S_1) \cup V(S_2) \subseteq V(S_1 \cdot S_2)$. Conversely suppose $a \in V(S_1 \cdot S_2)$ but $a \notin V(S_1)$; it suffices to show that $a \in V(S_2)$. Now $a \notin V(S_1) \Rightarrow$ there is some $f \in S_1$ so that $f(a) \neq 0$. Suppose that $g \in S_2$. Then $a \in V(S_1 \cdot S_2)$ $\Longrightarrow 0 = f(a)\, g(a)$. Since $f(a) \neq 0$ we must have $g(a) = 0$, and since $g \in S_2$ is arbitrary we have $a \in V(S_2)$.

(Top 3*) Claim $\bigcap_\alpha V(S_\alpha) = V(\bigcup_\alpha S_\alpha)$. ■

In zariski-topology.pdf we show that the family of Zariski open subsets is properly contained in the family of metric open subsets.

8. Continuity & Bases

Def. $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ topological spaces. Then $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous if for each V open in Y the set $f^{-1}[V]$ is open in X.

(Compatible with the def. for metric spaces) [Sutherland, Prop. 8.3]

Sutherland, Prop. 8.4 A composite of continuous functions is continuous.

Proof. $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$ continuous.

If W is open in Z, then $g^{-1}[W]$ is open in Y since g is continuous. Likewise, the set $f^{-1}[g^{-1}[W]]$ is open in X. Since the latter is $(g \circ f)^{-1}[W]$, $g \circ f$ is continuous. ∎

Sutherland, Prop. 8.6 (a) The identity map $1_X : (X, \mathcal{T}_X) \to (X, \mathcal{T}_X)$ is continuous.

(b) ~~Any~~ Every $f : (X, \text{discrete}) \to (Y, \mathcal{T}_Y)$ is continuous.

(c) A constant map is continuous.

(d) Every $f : (X, \mathcal{T}_X) \to (Y, \text{indiscrete})$ is continuous.

(e) Every $f : (\phi, \{\phi\}) \to (X, \mathcal{T}_X)$ is continuous.
(empty set is a topological space!)

<u>Bases = Topologies generated by families of subsets</u>

<u>Observation</u> An intersection of topologies on X is a topology on X.

If $\mathcal{O}$ is a family of subsets of X, let $\mathcal{T}(\mathcal{O})$ = intersection of all topologies $\mathcal{T}$ on X such that $\mathcal{O} \subseteq \mathcal{T}$. (Non empty – e.g. $\mathcal{T}$ = discrete).

<u>Proposition</u> $\mathcal{T}(\mathcal{O})$ consists of ϕ, X and arbitrary unions of finite intersections of sets in $\mathcal{O}$.

<u>Proof.</u> All such sets lie in $\mathcal{T}(\mathcal{O})$, so we need to show the conditions define a topology.

The only non-immediate part is to show the family described in the Proposition is closed under intersections (note that "a union of unions is a union" GERTRUDE STEIN PROPERTY)

But $\left(\bigcup_\alpha A_{\alpha_1} \cap \dots \cap A_{\alpha_{k(\alpha)}}\right) \cap \left(\bigcup_\beta A_{\beta_1} \cap \dots \cap A_{\beta_{\ell(\alpha)}}\right)$

$= \bigcup_{\alpha,\beta} A_{\alpha_1} \cap \dots \cap A_{\alpha_k} \cap A_{\beta_1} \cap \dots \cap A_{\beta_\ell}$. ■

$\boxed{(X, \mathcal{T}) \text{ top space}}$

Def. $\mathcal{B} \subseteq \mathcal{T}$ is a base for $\mathcal{T}$ if every open set is a union $\cup U_\alpha$ with $U_\alpha \in \mathcal{B}$.

Examples (X,d) metric and $\mathcal{B} = N_\varepsilon(x)$ where $x \in X$ and $\begin{cases} \varepsilon > 0 \\ \varepsilon = 1/n \text{ some } n. \end{cases}$

$(X, \text{discrete})$ $\mathcal{B} = \text{all } \{x\}$.

~~Another~~ One useful property of bases:

Sutherland, Prop. 8.12 $\mathcal{B}$ = base for $\mathcal{T}_Y$. Then $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous $\Longleftrightarrow$ $f^{-1}[V]$ is open for all V in $\mathcal{B}$.

PROOF: ($\Rightarrow$) Trivial since we know $V \in \mathcal{T}_Y \Rightarrow f^{-1}[V] \in \mathcal{T}_X$ and $\mathcal{B} \subseteq \mathcal{T}_Y$.

($\Leftarrow$) $V \in \mathcal{T}_Y \Rightarrow V = \bigcup_\alpha U_\alpha$ for $U_\alpha \in \mathcal{B}$.

Hence $f^{-1}[V] = f^{-1}[\cup_\alpha U_\alpha] = \cup_\alpha f^{-1}[U_\alpha]$, which is a union of open sets and hence is open. ∎

Note: If $\mathcal{T} = \mathcal{T}(\mathcal{A})$, say $\mathcal{A}$ is a sub-base for $\mathcal{T}$.

Homeomorphisms, Part A

$$f: (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y) \text{ s.t.}$$

f is 1-1 onto, f and f^{-1} both continuous.

(See Additional Exercise 8.2 for more)

Examples: (a,b) homeo (c,d)
$[a,b]$ homeo $[c,d]$
$(-1,1)$ homeo $\mathbb{R}$

First two: $f(x) = c + \dfrac{(d-c)}{(b-a)}(x-a)$ PRE CALC!

Last one: $f(x) = \dfrac{x}{1-|x|}$ OR $\tan \frac{\pi}{2} x$

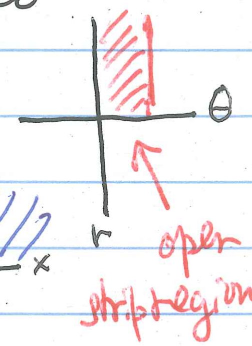
∞ DIFFERENTIABLE

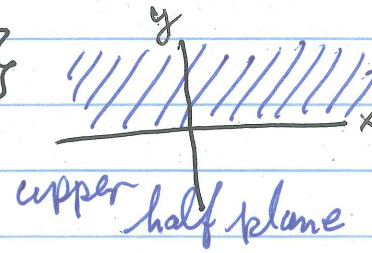
<u>Homeomorphisms, Part B</u> Geometrically, homeomorphic subsets of $\mathbb{R}^n$ resemble each other in some non-rigid sense.

For example, think about polar coords

$$x = r\cos\theta \qquad y = r\sin\theta$$

These define a homeo from $\{(r,\theta) \in \mathbb{R}^2 \mid r>0,\ 0<\theta<\pi\}$ to $\{(x,y) \in \mathbb{R}^2 \mid y>0\}$





Inverse map:

$$r = \sqrt{x^2+y^2}, \quad \theta = \arccos \frac{x}{\sqrt{x^2+y^2}}.$$

Go to <u>intro2topA-08.pdf</u> and <u>intro2topA-08a.pdf</u> for more.

The geometric behavior of homeomorphisms is behind the frequently repeated description of topology as a "rubber sheet geometry."