9. Some concepts in Topological Spaces

<u>Main purpose</u> Modify various concepts and results from Chapter 6 so they are meaningful for topological spaces.

<u>Two major differences</u> $(X, \mathcal{T})$ top space

(1) If $\mathcal{T}$ does not come from a metric, one point subsets are not necessarily closed. (Sierpiński topology, etc.)

(2) In a metric space, if $x \in X$ then there are open subsets $U_1 \supseteq U_2 \supseteq \dots$ such that $\{x\} = \bigcap_{n=1}^{\infty} U_n$: Take $U_n = N_{1/n}(x)$.

They arise naturally in some contexts

There are important examples of topological spaces which do not have this property.

~~[illegible]~~

~~[illegible]~~

<u>Example for (2)</u> [maybe not the most natural, but simple to work with].

X = uncountable set, say $\mathbb{R}$ or $\mathbb{C}$, with $\mathcal{T}$ = cofinite topology. — If we have open sets $U_1 \supseteq U_2 \ldots$ with $x \in U_i$ for all i, then $X - \bigcap_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} X - U_i = \bigcup_{i=1}^{\infty} F_i$ where F_i is finite, so $X - \bigcap_{i=1}^{\infty} U_i$ is countably infinite and hence $\bigcap_{i=1}^{\infty} U_i \neq \{x\}$.

The preceding two facts have the following significance:

(1) If $f : X \to Y$ is continuous, then the "level sets" $f^{-1}[\{y\}]$ need not be closed.

MAJOR POINT

(2) We cannot use ordinary convergent ∞ sequences when trying to prove results about topological spaces which might not come from metric spaces.

NOTE: There is a more general notion of (Moore-Smith) convergence which works in arbitrary topological spaces, but it is not as easy to work with in most cases, and it is much less widely used than ordinary convergence in metric spaces.

this is a little subjective

For example, M-S convergence is not covered in most graduate level topology courses, and has not been for several decades.

Sutherland, Prop. 9.5 $f:(X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous $\Leftrightarrow$ for each closed subset $F \subseteq Y$, the inverse image $f^{-1}[F]$ is closed in X.

Proof. Suppose f is continuous. Then F closed in $Y \Rightarrow Y - F$ open in $Y \Rightarrow$ $f^{-1}[Y-F] = X - f^{-1}[F]$ open in $X \Rightarrow$ $f^{-1}[F]$ closed in X. Conversely, if the condition in the proposition holds, then U open in $Y \Rightarrow Y - U$ ~~open~~ closed in $Y \Rightarrow$ $f^{-1}[Y-U] = X - f^{-1}[U]$ closed in $X \Rightarrow$ $f^{-1}[U]$ open in X. ∎

Here is the main thing we need:

Def. X top space, $A \subseteq X$. Then $p \in X$ is a limit point of $A \Leftrightarrow$ for each open set U containing p we have

$$(U - \{p\}) \cap A \neq \emptyset.$$

<u>CLAIM</u> For metric spaces, this is equivalent to the property in the previous definition.

$$\boxed{\text{For each } \varepsilon > 0,\ (N_\varepsilon(p) - \{p\}) \cap A \neq \emptyset}$$

<u>NEW ⟹ OLD</u> $N_\varepsilon(p)$ is an open set containing p

<u>OLD ⟹ NEW</u> Given U, choose $\delta > 0$ so that $N_\delta(p) \subseteq U$. Then $(N_\delta(p) - \{p\}) \cap A \neq \emptyset$ implies $(U - \{p\}) \cap A \neq \emptyset$.

<u>Modified counterpart to Sutherland, 6.14</u>

$A \subseteq X$ top space $\Rightarrow$

(ii) A is closed in $X \Leftrightarrow L(A) \subseteq A$

(iii) $A \cup L(A)$ is closed in X

(i)* If $\{p\}$ is closed for all $p \in X$, then $L(A)$ is closed in X.

The proofs of (ii) & (iii) from Chapter 6 go through if we replace the metric neighborhoods $N_\varepsilon(p)$ with arbitrary open sets containing p.

However, we need the following new proof for (i)*:

It suffices to show $L(L(A)) \subseteq L(A)$.

Suppose $p \in L(L(A))$. Let U be open with $p \in U$. Then there is some point $q \in (U - \{p\}) \cap L(A)$. By hypothesis $U - \{p\}$ is open, so $q \in L(A) \Rightarrow$ there is some point $q' \in (U - \{p, q\}) \cap A \subseteq (U - \{p\}) \cap A$. ∎

The whole discussion of closures now goes through for topological spaces (taking earlier points (1) and (2) into account).

<u>Interiors</u> Redefine $\mathrm{Int}(A, X)$ as all $a \in A$ such that there is some open set U with $a \in U \subseteq A$.

As before, this reduces to the earlier definition for metric spaces. The main adjustment needed involves the proof that $\mathrm{Int}(A, X) =$ largest open set $\subseteq X$.

Changes needed (i) U open and $U \subseteq A \Rightarrow U \subseteq \text{Int} A$. — No need to talk about N_ε's.

(ii) Prove $\text{Int}(A, X)$ is open. If $a \in \text{Int}(A, X)$ and $a \in U_a$ open with $U_a \subseteq A$, then $b \in U_a \Rightarrow b \in \text{Int} A$, so $U_a \subseteq \text{Int}(A)$. Hence

$\text{Int} A = \cup \{a\} \subseteq \cup U_a \subseteq \text{Int} A$ as before. ∎

so A is a union of open sets.

Modified Sutherland, Prop. 6.24 [corrected]
$x \in X$, $A \subseteq X$. Then $x \in \text{Bdy}(A, X) \Leftrightarrow$ for all open sets U with $x \in U$, both $A \cap U$ and $(X - A) \cap U$ are nonempty. ∎

[Again, replace $N_\varepsilon(x)$'s with U's.]

Sutherland, Def. 9.22 $x \in N \subseteq X$. Say N is a neighborhood of x if there is an open set U in X such that $x \in U \subseteq N$.

$\Rightarrow$ Not necessarily open! For example, $[a-h, a+h]$ is a closed neighborhood of a in $\mathbb{R}$ for each $h > 0$. ← N can be neither open nor closed

10. Subspaces and Product Spaces

More generalizations from metric to topological spaces.

<u>Def.</u> $(X, \mathcal{T})$ topological space, $A \subseteq X$. Then $\mathcal{T}|A$ ($\mathcal{T}$ restricted to A) is all sets $V \cap A$, where V is open in X.

We should check this is a topology, and if $\mathcal{T}$ comes from a metric d, then $\mathcal{T}|A$ comes from $d|A \times A$. First part is Exe. 10.2, the second is Exc. 10.4.

<u>Sutherland, 10.4–10.6</u>

(i) The inclusion $j = j_{A \subseteq X} : A \to X$ is continuous.

(ii) $f: X \to Y$ continuous $\Rightarrow f|A = f \circ j : A \to X$ is continuous.

restriction of f to A

(iii) If $f[X] \subseteq B$, then the function $g: X \to B$, $g(x) = f(x)$, is continuous.

<u>Proofs.</u> (i) U open in $X \Rightarrow j^{-1}[U] = U \cap A$.

(ii) $f|A$ is a composite of cont. fcns.

(iii) Suppose V is open in B, and write $V = B \cap U$ where U is open in Y. Then $f^{-1}[U]$ is open. However, $f[X] \subseteq B \Rightarrow$ $f^{-1}[U] = f^{-1}[U \cap B] = g^{-1}[V]$, so the latter is open in X. ■

<u>Gertrude Stein Rules</u> (An $\boxed{*}$ of a $\boxed{*}$ is a $\boxed{*}$).

"WILD CARD"

B is $\begin{Bmatrix}\text{open}\\ \text{closed}\end{Bmatrix}$ in A (with respect to subspace topology) and A is $\begin{Bmatrix}\text{open}\\ \text{closed}\end{Bmatrix}$ in $X \Rightarrow B$ is too.

<u>Proof.</u> Write $B = C \cap A$ where C is $\begin{Bmatrix}\text{open}\\ \text{closed}\end{Bmatrix}$ in X. Since a finite intersection of $\begin{Bmatrix}\text{open}\\ \text{closed}\end{Bmatrix}$ sets is $\begin{Bmatrix}\text{open}\\ \text{closed}\end{Bmatrix}$, this means B is $\begin{Bmatrix}\text{open}\\ \text{closed}\end{Bmatrix}$ in X. ■

<u>PRODUCT TOPOLOGIES</u> $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ top spaces. Then the product topology is generated by all sets $U \times V$, where $\begin{cases} U \in \mathcal{T}_X \\ V \in \mathcal{T}_Y \end{cases}$.

If $\mathcal{T}_X$ and $\mathcal{T}_Y$ come from metrics, this is the d_∞ product metric topology by definition. Hence <u>all</u> d_p metrics define the product topology on $X \times Y$. ($p = 1, 2, \infty$)

<u>Note</u> Not every set in the product top. is a finite union of rectangular open sets. See <u>intro2top A-10.pdf</u> for more.

<u>Products and continuous functions</u>

$\pi_X, \pi_Y : X \times Y \longrightarrow X, Y$ are continuous

$$\pi_X^{-1}[U] = U \times Y, \quad \pi_Y^{-1}[V] = X \times V,$$

so inverse images of open sets are open.

<u>Sutherland, Prop 10.11 modified</u> X, Y, W top spaces, $f : W \longrightarrow X$ and $g : W \longrightarrow Y$ cont. Define a set theoretic function $h : W \longrightarrow X \times Y$ by $h(w) = (f(w), g(w))$. Then h is continuous.

Proof. (1) Show inverse images of basic open subsets are open. But $h^{-1}[U \times V] =$

$$f^{-1}[U] \times g^{-1}[V].$$ If f, g continuous, this is a product of open sets and hence is open.

(2) General case. We can write $D = \cup U_\alpha \times V_\alpha$ if D is open in $X \times Y$, and

$$h^{-1}[D] = \bigcup_\alpha h^{-1}[U_\alpha \times V_\alpha].$$ By (1) the RHS is a union of open subsets and hence the union is open in W. ∎

Sutherland, 10.14 Given (x_0, y_0) in $X \times Y$, the slice inclusions $i(x_0): Y \longrightarrow X \times Y$, $y \longrightarrow (x_0, y)$ and $j(y_0): X \longrightarrow X \times Y$, $x \longrightarrow (x, y_0)$ define homeomorphisms onto their images.

Proof. The maps are 1-1 onto, and they are continuous because $\{f, g\} = \{\text{identity}, \text{constant}\}$ in each case. Check directly that inverses are given by projection onto $\left\{\begin{matrix} Y \\ X \end{matrix}\right\}$ for $\left\{\begin{matrix} i \\ j \end{matrix}\right\}$. ∎

More precise version

Let $i' : Y \longrightarrow \{x_0\} \times Y$ be induced by
$j' : X \longrightarrow X \times \{y_0\}$ i and j.

Then i' + j' are 1-1, onto, and continuous.
The continuous inverses are the mappings

$\pi_Y \mid \{x_0\} \times Y$
$\pi_X \mid X \times \{y_0\}$
respectively. ∎

Sutherland, 10.19 $f: X \to Y$ cont. Γ_f = graph of f = all $(x,y) \in X \times Y$ so that $y = f(x)$. Then Γ_f is homeomorphic to X.

Proof: Define $h: X \to X \times Y$ s.t. $\pi_X h = id_X$, $\pi_Y h = f$, so h is continuous. It is 1-1 since $h(x) = h(x') \Rightarrow x = \pi_X h(x) = \pi_X h(x') = x'$. Let $h_0 : X \to \Gamma_f$ be the induced 1-1 onto continuous map, and let $k_0 : \Gamma_f \to X$ be $\pi_X | \Gamma_f$. Then k_0 is a continuous inverse to h_0. ∎

Comment: If X and Y come from metric spaces, then $\Gamma_f \subseteq X \times Y$ is closed. Generalizations to top spaces are discussed in Chapter 11.

Read Sutherland 10.20: A useful tool for showing subsets in $X \times Y$ are open in some cases.

11. The Hausdorff Condition

<u>Hausdorff Separation Property in $(X, \mathcal{T})$</u>

Given $x \neq y$ there are open subsets U, V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

<u>Example</u> (X, d) metric, $r = d(x,y)$.

$U = N_{r/2}(x)$, $V = N_{r/2}(y)$

If $z \in U \cap V$ then $r = d(x,y) \leq d(x,z) + d(z,y)$

$< \frac{r}{2} + \frac{r}{2} = r$

CONTRADICTION.

<u>Example</u> Cofinite topology on infinite X is not Hausdorff: U, V open nonempty $\Rightarrow$ $U \cap V$ is infinite.

<u>Important properties in the exercises</u>:

EXERCISE #

(11.2) ① X hausdorff $\Rightarrow$ every one point subset is closed

(11.7) ② X Hausdorff $\Leftrightarrow$ diagonal $\Delta_X = \{(x_1, x_2) \in X \times X \mid x_1 = x_2\}$ is closed in $X \times X$

(11.10) ③ $f_0, f_1 : X \to Y$ continuous + Y Hausdorff $\Rightarrow$ set of points in X where $f_0(x) = f_1(x)$ is closed in X.

Prop. $A \subseteq X$ & X Hausdorff $\Rightarrow A$ Hausdorff.

Proof $a_1 \neq a_2$ in A. There are open sets U_i in X such that $a_i \in U_i$ and $U_1 \cap U_2 = \emptyset$. Let $V_i = A \cap U_i$. Then $a_i \in V_i$ and $V_1 \cap V_2 = \emptyset$.

Prop. If X and Y are Hausdorff, so is $X \times Y$.

Proof Exercise 11.4. ∎

Regular and normal spaces

Theorem (X, d) metric E & F disjoint closed subsets. Then there are open sets U, V s.t. $E \subseteq U$, $F \subseteq V$ and $U \cap V = \emptyset$. (NORMAL SPACE)

Proof. Let $f(x) = \dfrac{d(x,E)}{d(x,E) + d(x,F)}$

The denominator is nonzero, for $E \cap F = \emptyset \Rightarrow$ $x \notin E$ or $x \notin F$, so either $d(x,E) > 0$ or $d(x,F) > 0$.

f is continuous, $f = 0$ on E
$f = 1$ on F.

Take $U = f^{-1}[(-\infty, \frac{1}{2})]$, $V = f^{-1}[(\frac{1}{2}, +\infty)]$. ∎