12. Connected Spaces

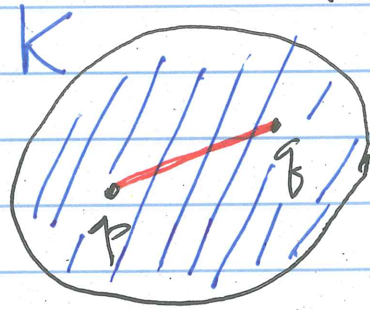
Intermediate Value Property

f : interval in $\mathbb{R} \longrightarrow \mathbb{R}$ continuous

u, v in domain such that $f(u) < f(v)$,
c such that $f(u) < c < f(v)$. THEN
there is some w in the domain s.t. $f(w) = c$.

PROBLEM. Find simple but general criteria $\underbrace{}_{\boxed{\text{for metric \& top spaces}}}$
under which the conclusion is true.

EXAMPLE $\left\{\begin{matrix}\text{closed} \\ \text{open}\end{matrix}\right\}$ disk, $x^2+y^2 \left\{\begin{matrix}\leq \\ <\end{matrix}\right\} 1$



square $0 \leq x, y \leq 1$.

Examples of convex sets:
$p, q \in K \Rightarrow$ the closed
segment $p + t(q-p) \in K$, $0 \leq t \leq 1$

"no dents or holes"

CLAIM The Intermediate value property is true for continuous functions on K.

First, let's verify the three sets are convex.

Disks $|p|, |q| \left\{ \begin{matrix} \le \\ < \end{matrix} \right\} 1 \Longrightarrow$ $\boxed{\text{Euclidean norm!}}$

$|p + t(q-p)| = |tq + (1-t)p| \le$

$|t| \cdot |p| + |1-t||q| \underset{t,\, 1-t \ge 0}{=\!=} t|p| + (1-t)|q|$

$\left\{ \begin{matrix} \le \\ < \end{matrix} \right\} t \cdot 1 + (1-t) \cdot 1 = 1.$

$\boxed{\text{Note that } p + t(q-p) = tq + (1-t)p}$

Square ~~$w = x$ ang~~ Write $p = (p_1, p_2)$
$q = (q_1, q_2)$

Both in $[0,1] \times [0,1]$. Then $tq + (1-t)p =$

$(tq_1 + (1-t)p_1, tq_2 + (1-t)p_2)$. Then

$0 \le p_i, q_i \le 1 \Rightarrow 0 \le tq_i + (1-t)p_i \le$
$0 \le t \le 1$
$t + (1-t) = 1. \blacksquare$

Now, prove the claim $f(w) < f(v)$

with $f: K \longrightarrow \mathbb{R}$ continuous.

Let $h(t) = f(tv + (1-t)w)$, $t \in [0,1]$.

Then h is continuous, $h(0) = f(u)$
$h(1) = f(v)$.

Therefore there is some $t^* \in (0,1)$ such that $h(t^*) = c$. But now

$$c = f(\underbrace{t^* v + (1-t^*) u}_{\text{lies in } K \text{ by convexity.}})$$

ISSUES 1: We have only shown that if intervals have the Intermediate Value Property, then so do convex sets

See Sutherland Thm. 4.35

2. We still want a simple criterion, expressible in terms of the topology on a set.

Def $(X, \mathcal{T})$ is connected if the only subsets U which are both open and closed (clopen!) are ϕ and X. Note U clopen in $X \Rightarrow$ so is $X - U$.

Example A discrete space with ≥ 2 points is not connected. ($\{x\}$ is clopen, all $x \in X$).

2. If $J \subseteq \mathbb{R}$ is such that $u < v$ in $J \Rightarrow (u, v) \subseteq J$, then J is connected.

Proof of #2. Suppose $\phi \neq U$, $J \neq U$ where $U \subseteq J$ is clopen. Then $V = J - U$ is also a nonempty proper clopen subset. Let $u \in U$ and $v \in V$. Without loss of generality, $u < v$. (if $u > v$, reverse roles of u, v and U, V in the discussion which follows).

By hypothesis, (u, v) and $[u, v] \subseteq J$.

Let $b^* = \text{l.u.b. } U \cap [u, v]$ (nonempty, for u lies in this set).

(1) Suppose $b^* < v$. Then $b^* = \text{l.u.b. } U \cap [u, v]$ and $[u, v] \subseteq J \Rightarrow (b^*, v] \subseteq V$. Since V is closed, we also have $[b^*, v] \subseteq V$ so that $b^* \in V$. If $b^* = v$ the same conclusion holds by assumption.

Hence $b^* > u$, since $u \in U$.

(2) **CLAIM:** There is a sequence of points $x_n \in U$ (n suff. large) such that

$$b^* - \frac{1}{n} < x_n \leq b^*$$

(Exists since b^* is a least upper bound)

Then $\lim_{n\to\infty} x_n = b^*$. Since $U \cap [u,v]$ is closed, we have $b^* \in U$.

① & ② contradict each other. The source of the problem is the assumption that J is a union of two nonempty disjoint clopen subsets. Hence this is false and J is connected. ∎

<u>Converse Statement</u> If $J \subseteq \mathbb{R}$ with $u < v$ in J and $u < x < v$ with $x \notin J$, then J is not connected.

<u>Verification</u> $(-\infty, x) \cup (x, +\infty) = \mathbb{R}$

disjoint open sets.

Hence J is the union of the nonempty disjoint subsets $(-\infty, x) \cap J$, $(x, +\infty) \cap J$ (since $x \notin J$).

We can now prove an abstract form of the Intermediate Value Theorem.

Sutherland, Prop. 2.11 $f: X \to Y$ continuous, X connected $\Rightarrow f[X]$ connected.

Proof. Prove contrapositive: If $f[X]$ is not connected, then neither is X.

Suppose $W \subseteq f[X]$ is a nonempty, clopen proper subset. Write $W = f[X] \cap U$, U open in Y

$f[X] \cap E$, E closed in Y.

Then $f^{-1}[W] = f^{-1}[f[X] \cap U] = f^{-1}[f[X]] \cap f^{-1}[U]$

$= f^{-1}[U]$, open in X

also $= f^{-1}[E]$ closed in X

Finally, since W is a nonempty proper subset of $f[X]$ we have $x_1, x_2 \in X$ such that $f(x_1) \in W$ and $f(x_2) \notin W$. Hence $f^{-1}[W]$ is a nonempty proper subset of X, so X is not connected. ∎

Corollary If $Y = \mathbb{R}$ and $f(x_1) < f(x_2)$, then $[f(x_1), f(x_2)] \subseteq f[X]$. ($X$ connected).

(Intermediate Value Property). ∎

Recognizing connected (sub) spaces

Sutherland, 12.16 $A_\alpha \subseteq X$ connected all α. Suppose for all α, β we have $A_\alpha \cap A_\beta \neq \emptyset$. Then $\cup A_\alpha$ is connected.

We might as well replace X by $Y = \cup A_\alpha$.

Proof. Suppose Y is not connected, and let $W \subseteq Y$ be nonempty, clopen ($\Rightarrow Y - W$ clopen)

For each α, $W \cap A_\alpha$ is clopen in $A_\alpha \Rightarrow$ either $W \cap A_\alpha = \emptyset$ or $W \cap A_\alpha = A_\alpha$.

so $A_\alpha \subseteq Y - W$ so $A_\alpha \subseteq W$.

We might as well assume $A_{\alpha_0} \subseteq W$ (otherwise switch the roles of W and $Y - W$).

Since $A_\alpha \cap A_{\alpha_0} \neq \emptyset$ and $A_\alpha \cap A_{\alpha_0} \subseteq W$, we must have $W \supseteq A_\alpha$ ($A_\alpha \subseteq Y - W \Rightarrow A_\alpha \cap A_0 \cap W = \emptyset$).

Hence $\bigcup_\alpha A_\alpha \subseteq W$, so $W = Y$. ∎

Thm. 12.18 X, Y connected $\Rightarrow$ so is $X \times Y$.

Proof. X is homeo to $X \times \{y_0\} \Rightarrow$ latter connected

Y is homeo to $\{x_0\} \times Y \Rightarrow$ $\boxed{\text{DITTO}}$

Hence for each $x \in X$ the set

$$A_x = (X \times \{y_0\}) \cup (\{x\} \times Y) \text{ is connected.}$$

THEY MEET AT (x, y_0).

We have $X \times \{y_0\} \subseteq A_x \cap A_{x'}$, so

$$X \times Y = \bigcup_{x \in X} A_x \text{ is connected.} \blacksquare$$

Cor. A. J_1, J_2 intervals in $\mathbb{R} \Rightarrow J_1 \times J_2$ conn.

Cor. B. $X_1, \dots, X_n$ conn. $\Rightarrow \prod_{i=1}^{n} X_i$ conn.

Prop. 12.19 A connected $\subseteq X$ and $A \subseteq B \subseteq \overline{A} \Rightarrow B$ connected.

Proof. Without loss of generality, we may assume $B = X$.

Suppose B is not connected, and let U be a nonempty, clopen, proper subset of B, so that $V = X - U$ is also a nonempty, clopen, proper subset.

We then have $A = (A \cap U) \cup (A \cap V)$ where $A \cap U$ and $A \cap V$ are clopen in A. Hence $\{A \cap U, A \cap V\} = \{A, \Phi\}$ (in no particular order). Let's say (without loss of generality) that $A \cap U = A$, so that $A \subseteq U$. Now U is closed in B, so $B = \overline{A} \subseteq U$ and hence $B = U$. This is a contradiction. — The source is the assumption that B is not connected, so the latter must be false. Hence B is connected. ■

<u>Cor.</u> If E is a subset of S^1, then $N_1(0; \mathbb{R}^2) \cup E$ is connected. — This means that $\mathbb{R}^2$ has many more connected subsets than $\mathbb{R}$ has (in fact, it's the same as the cardinal number of subsets of $\mathbb{R}^2$ itself). In fact the subsets in the corollary are all <u>convex</u>.

Path/Arcwise Connected Spaces

X — Given $x \neq y$ in X, can find $\gamma : [a,b] \to X$ continuous so that $\gamma(a) = x$, $\gamma(b) = y$.

$\boxed{\text{Say } \gamma \text{ is a continuous curve (or ARC) joining } x \text{ to } y \text{ (or } x \text{ and } y\text{).}}$

Prop. (1) Arcwise connected $\Rightarrow$ connected

(2) X, Y arcwise connected $\Rightarrow$ so is $X \times Y$.

Proof. (1) Fix $p \in X$. If $x \in X$, let γ be a curve joining p to x, and let $\Gamma_x = $ image γ. Then Γ_x is connected and $p \in \Gamma_x$ all x, so

$$X = \bigcup_{x \in X} \{x\} \subseteq \bigcup_{x \in X} \Gamma_x \subseteq X \Rightarrow X = \bigcup_{x \in X} \Gamma_x .$$

By a previous result, RHS is connected $\Rightarrow$ so is X. ∎

(2) Let (x_1, y_1) & $(x_2, y_2) \in X \times Y$.

Let ~~$\gamma_i : [a_i, b_i]$ join~~ $\left\{ \begin{array}{l} \gamma_X : [a_X, b_X] \to X \\ \gamma_Y : [a_Y, b_Y] \to Y \end{array} \right\}$

join $\left\{ \begin{array}{l} x_1 \text{ to } x_2 \\ y_1 \text{ to } y_2 \end{array} \right\}$, and let h_X, h_Y be linear $[0,1] \to \begin{array}{l} [a_X, b_X] \\ [a_Y, b_Y] \end{array}$

functions with $\begin{Bmatrix} h_X(0)=a_X, & h_X(1)=b_X \\ h_Y(0)=a_Y, & h_Y(1)=b_Y \end{Bmatrix}$.

Then
$$\begin{tikzcd} {[0,1]} \arrow[r, "(h_X,h_Y)"] \arrow[dr, "\beta"'] & {[a_X,b_X]\times[a_Y,b_Y]} \arrow[d, "\gamma_X\times\gamma_Y"] \\ & X\times Y \end{tikzcd}$$

β joins (x_1,y_1) to (x_2,y_2). ∎

<u>Prop. 12.25.</u> If U is nonempty and open in $\mathbb{R}^n$ $(n\geq 1)$ with U connected, then U is arcwise connected.

<u>Proof.</u> Given any space X, write $x\sim y \iff$ there is a cont curve $\gamma:[a,b]\longrightarrow X$ joining x and y. <u>CLAIM:</u> This is an equivalence relation.

$\boxed{x\sim x}$ Take γ = constant curve at x.

$\boxed{x\sim y \Rightarrow y\sim x}$ Given γ as above, define

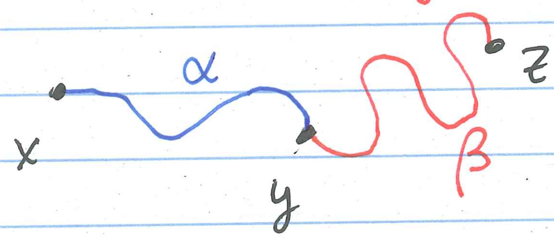
$$\gamma^*: [-b,-a] \xrightarrow{(-1)} [a,b] \xrightarrow{\gamma} X,$$

so $\gamma(-b)=y$ & $\gamma(-a)=x$.

$\boxed{x\sim y \;\&\; y\sim z \Rightarrow x\sim z}$ Given $\begin{cases}\alpha:[a,b]\to X\\ \beta:[c,d]\to X\end{cases}$

$\alpha(a)=x$, $\alpha(b)=y=\beta(c)$, $\beta(d)=z$.

CONCATENATE (string together) THE CURVES.



$\gamma=\alpha+\beta$

$\gamma:[a,b+d-c]\to X$ $\quad\gamma(t)=\alpha(t)$ if $t\le b$

$\gamma(t)=\beta(\underset{c+t-b}{\cancel{\quad}})$ if $t\ge b$. These fit together

because $\underset{t=b}{\alpha(b)}=y=\beta(c+\overset{t=b}{\cancel{b-b}})$.

Note that $\gamma(b+d-c)=\beta(c+(b+d-c)-b)=\beta(d)=z$

The equivalence classes are open, for if

in $\mathbb{R}^n$ $N_\varepsilon(p)\subseteq U$, then $N_\varepsilon(p)\subseteq$ equivalence class of p.

Equivalence classes are pairwise disjoint, so if there is more than one equivalence class, then each equivalence class is an open subset, and likewise for its complement (which is a union of eq. classes). Since U is conn., only one eq. class, and this implies U is arcwise conn. ∎

Application — Showing Spaces are not homeomorphic

Prop. $f : X \to Y$ homeomorphism & $x \in X$ $\Longrightarrow$ f maps $X - \{x\}$ homeomorphically to $Y - \{f(x)\}$.

Proof. Let U be open in $X - \{x\}$, so $U = W \cap (X - \{x\})$ where W is open in X. Then $f[U] = f[W] \cap f[X - \{x\}] = f[W] \cap Y - \{f(x)\}$, where $f[W]$ is open in Y since f is a homeo; hence $f[U]$ is open in $Y - \{f(x)\}$. Conversely, if V is open in $Y - \{f(x)\}$, then $V = W \cap (Y - \{f(x)\})$ where W is open in Y. Then $f^{-1}[V] = f^{-1}[W] \cap (X - \{x\})$; by continuity $f^{-1}[W]$ is open in X, so $f^{-1}[V]$ is open in $X - \{x\}$. Hence $f_0 : X - \{x\} \to Y - \{f(x)\}$, with $f_0(t) = f(t)$, is a homeomorphism. ∎

<u>Cor.</u> Same conclusion if $\{x\}$ is replaced by $\{x_1, \ldots, x_k\}$ <u>etc.</u> [induction]

<u>Distinguishing interval types</u>

$[a, b]$ — There are exactly two points in X such that $X - \{x\}$ is connected.

(a, b) — $X - \{x\}$ is never connected.

$[a, b)$ — There is a unique point in X such that $X - \{x\}$ is connected.

S^1 — For each x, $X - \{x\}$ is conn.

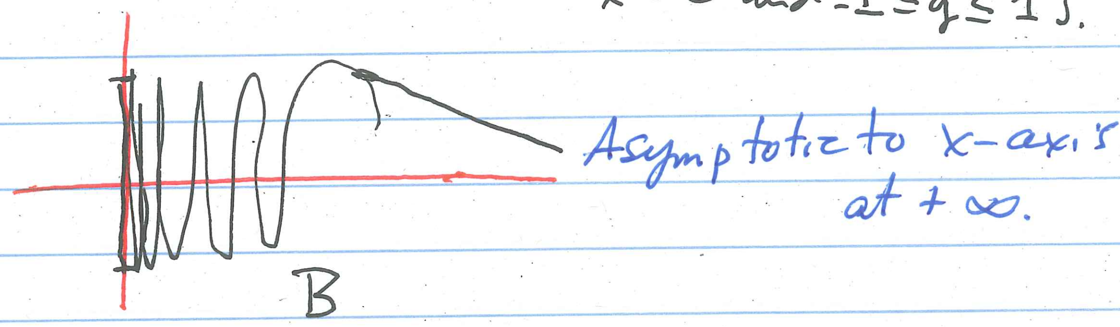
<u>PROOF.</u> $x = (\cos 2\pi s_0, \sin 2\pi s_0)$ some s_0.

Let $\gamma(t) = (\cos 2\pi(s_0 + t), \sin 2\pi(s_0 + t))$ $0 < t < 1$. Then image $\gamma = S^1 - \{x\}$, so the latter is connected.

<u>NO PROOF.</u> $S^1 -$ two pts. is not connected, but $\mathbb{R}^2 -$ two pts is connected.

Example

let $B = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0$ and $y = \sin\frac{1}{x}$ or
$x = 0$ and $-1 \leq y \leq 1\}$.



Then B is connected but not arcwise conn.
(no continuous curves in B joining $(0,0)$ to
points with $x > 0$).

See the second example on p. 65 of
graduate-level-classnotes.pdf
and further discussion of it on p. 66 for
an explanation.