13. Compact spaces

Extreme Value Property

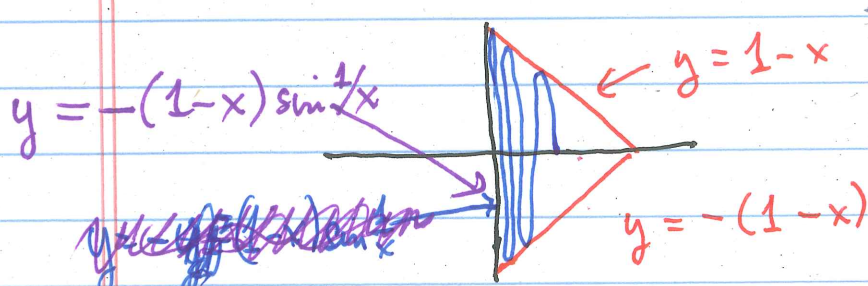
$f : [a,b] \longrightarrow \mathbb{R}$ continuous $\Longrightarrow$

f ~~has~~ takes a maximum and minimum value on $[a,b]$.

Examples ① $f(x) = x$ $\mathbb{R} \to \mathbb{R}$, no upper or lower bounds on the values $f(x)$.

② $f(x) = \dfrac{x}{1+|x|}$ $\mathbb{R} \longrightarrow \mathbb{R}$ l.u.b of all $f(x)$ is $+1$, g.l.b. is -1, but $f(x)$ is never ± 1.

③ $f(x) = -(1-x)\sin\frac{1}{x}$ on $(0, \frac{1}{\pi})$.



l.u.b $f(x) = +1$

g.l.b. $f(x) = -1$

but $f(x)$ is never ± 1.

Problem. Find general criteria under which the conclusion is true $\boxed{\text{on metric + top. spaces}}$

Let's see how one proves the Extreme Value Property

Heine-Borel-Lebesgue Theorem

Let $[a,b]$ be a closed interval, and suppose that for each $x \in [a,b]$ there is an open interval $(x-\delta_x, x+\delta_x)$. Then there is a finite number of points $x_1, \dots, x_m$ such that $[a,b] = \bigcup_{i=1}^{m} (x_i - \delta(x_i), x_i + \delta(x_i))$.

Proof. Let $J_x = (x-\delta_x, x+\delta_x)$, and say $y \in [a,b]$ is accessible if $[a,y]$ is contained in a union of finitely many J_x's.

CLAIM $y^* =$ L.U.B. of accessible points $\Longrightarrow$ $y^* = b$ (note that a is accessible). $\boxed{\text{THIS PROVES THM.}}$

(1) y^* is accessible. By def of L.U.B., there is an accessible point x^* in $(y^* - \delta_{y^*}, y^*]$

Regardless of whether $x^* < y^*$ or $x^* = y^*$, we see that y^* is accessible (add J_{y^*} to the finite collection of intervals).

② If $y^* < b$, then $y^* < z \leq b \Rightarrow z$ is not accessible because y^* is the L.U.B of accessible points.

③ On the other hand, every point in $J_{y^*} \cap [a,b]$ is also accessible. CONTRADICTION

The source of the problem is the hypothesis that $y^* < b$. — Hence y^* must be equal to b. ∎

Corollary 1. f is bounded on $[a,b]$.

Proof Let $\varepsilon > 0$, and take J_x's so that $t \in (x-\delta_x, x+\delta_x) \Rightarrow |f(t)-f(x)| < \varepsilon$.

By HBL, $[a,b] \subseteq J_{x_1} \cup \dots \cup J_{x_m}$.

Hence $f(x) \leq \max\{f(x_i)+\varepsilon\}$

$f(x) \geq \min\{f(x_i)-\varepsilon\}$. ∎

Corollary 2. f takes maximum and minimum values.

$$m = \underset{x}{GLB}\, f(x) \leq \underset{x}{LUB}\, f(x) = M.$$

We might as well assume $m > 0$; replace $f(x)$ by $g(x) = f(x) + C$ where $m + C > 0$.

① Suppose $f(x) < M$ always. Then

$$\frac{1}{M - f(x)}$$ is continuous but unbounded. (*)

② Suppose $m < f(x)$ always. Then

$$\frac{1}{f(x) - m}$$ is continuous but unbounded (*)

(*) In each case this is true because $\begin{Bmatrix} M - f(x) \\ f(x) - m \end{Bmatrix} > 0$ but for each ~~M~~ $K > 0$ we can find x such that $\begin{Bmatrix} M - f(x) \\ f(x) - m \end{Bmatrix} < \frac{1}{K}$.

Hence we can find x s.t. $\frac{1}{M - f(x)}$ or $\frac{1}{f(x) - m} > K$.

respectively

In both cases the first corollary yields a contradiction. The source is the assumption that $\begin{Bmatrix} f(x) < M \\ f(x) > m \end{Bmatrix}$ always, so this is false and $f(x)$ can be chosen to be either M or m. ∎

Putting Things into a more general setting

Want a comparable result with $[a,b]$ replaced by a closed bounded subset of $\mathbb{R}^n$, ~~and~~ but even more general than that.

Def. A topological space X is compact if for every family of open subsets $\mathcal{U} = \{U_\alpha\}$ such that $X = \bigcup_\alpha U_\alpha$, there is a finite subcollection $\{U_{\alpha_1}, \dots, U_{\alpha_k}\}$ such that $X = U_{\alpha_1} \cup \dots \cup U_{\alpha_k}$.

Say $\mathcal{U}$ = open covering, $\{U_{\alpha_1}, \dots, U_{\alpha_k}\}$ finite subcovering

Example Heine-Borel-Lebesgue $\Longrightarrow$ $[a,b]$ is compact.

We can now modify the proof of the Extreme Value Property so that it is valid if X is a compact space.

NEXT Which subsets of $\mathbb{R}^n$ are compact?

Sutherland, Prop. 13.10 (X, d) metric and $A \subseteq X$ is compact $\Rightarrow A$ is bounded.

Proof. Without loss of generality, $A = X$. Fix $\varepsilon > 0$ and take the open covering $\mathcal{U}$ whose elements are the sets $N_\varepsilon(x)$ for $x \in X$. Let $N_\varepsilon(x_1), \ldots, N_\varepsilon(x_k)$ be a finite sub-covering, and let $x_0 \in X$. Let $D =$ max ~~of~~ $d(x_0, x_i)$. Then $y \in X \Rightarrow y \in N_\varepsilon(x_i)$ some i $\Rightarrow d(x_0, y) \leq d(x_0, x_i) + d(x_i, y) <$

$$M + \varepsilon. \blacksquare$$

So $A \subseteq \mathbb{R}^n$ compact $\Rightarrow$ closed, bdd

Sutherland, Prop 13.12 $(X, \mathcal{T})$ Hausdorff and $A \subseteq X$ compact $\Rightarrow A$ closed in X.

Proof. Show $X - A$ is open in X.

Let $y \in X - A$, $z \in A$. Then there are disjoint open subsets $U_{y,z}$ & $V_{y,z}$ containing y and z respectively.

The sets $\{V_{y,z} \cap A\}$ form an open covering of A. Take a finite subcovering $V_{y,z_1} \dots V_{y,z_k}$ and let $U_y = U_{y,z_1} \cap \dots \cap U_{y,z_k}$. * Then $y \in U_y$ and

*See the drawing on p. 131 of Sutherland

V_{y,z_j} abbreviated

$$U_y \cap A \subseteq U_y \cap (\cup V_j) =$$

$$\cup (U_y \cap V_j) \subseteq \cup (U_{y,z_j} \cap V_{y,z_j}) =$$

$\cup$ empty sets $= \emptyset$. Hence

$$X - A = \bigcup_{y \notin A} U_y$$ must be open in X. ∎

To finish we must show that if $A \subseteq \mathbb{R}^n$ is closed & bounded, then it is compact. This can be done abstractly as follows.

<u>Sutherland, Prop. 13.20</u> If A is closed in X and X is compact, then A is also compact.

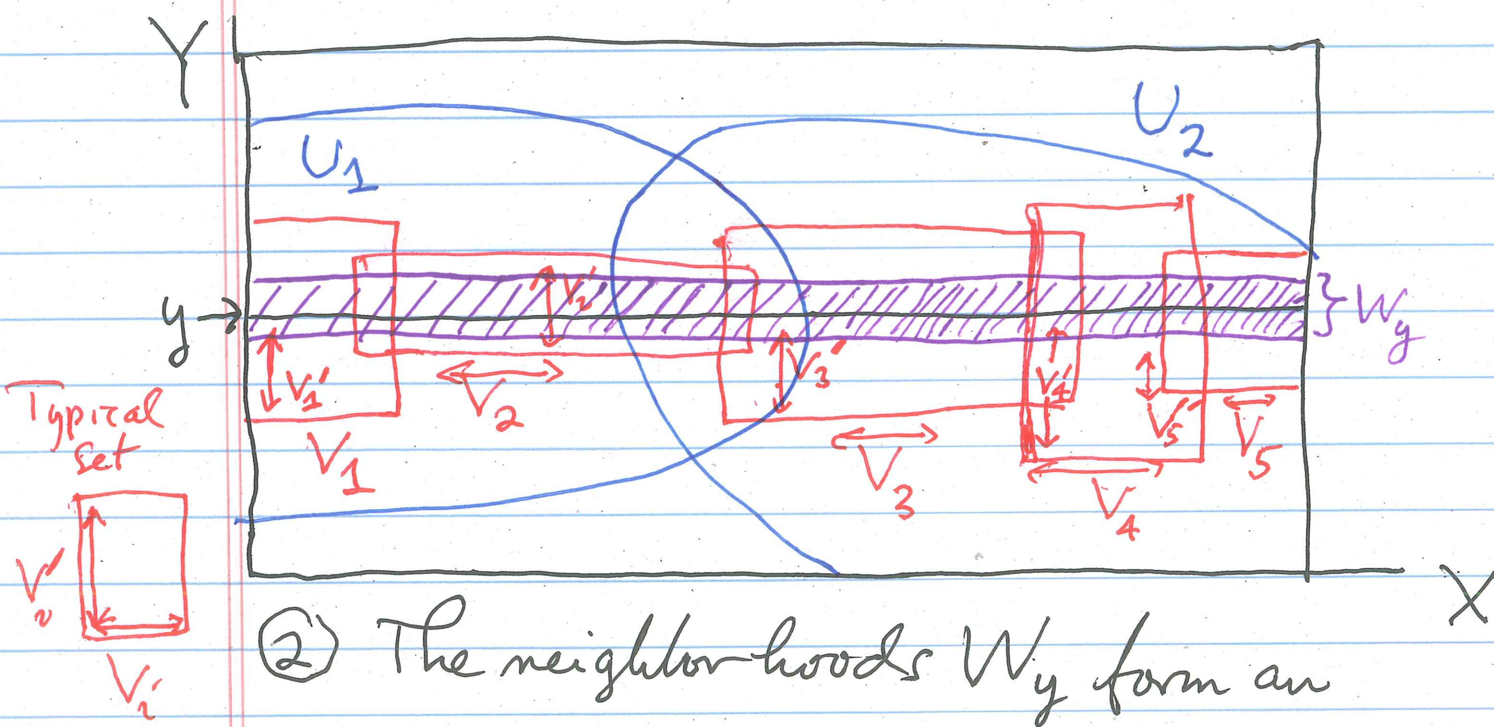
<u>Proof.</u> Let $\{U_\alpha\}$ be an open covering of A.
For each U_α, let $V_\alpha \subseteq X$ be open such that $U_\alpha = V_\alpha \cap A$. Let $\mathfrak{V}$ = all V_α's plus $X-A$.
Then $\mathfrak{V}$ is an open covering of $X \Rightarrow$ there is a finite subcovering $V_{\alpha_1}, \dots, V_{\alpha_k}, X-A$.
Then $U_{\alpha_1} \cup \dots \cup U_{\alpha_m} = (V_{\alpha_1} \cap A) \cap \dots \cap (V_{\alpha_m} \cap A)$
$= (\cup V_{\alpha_i}) \cap A$ Now $\cup V_{\alpha_i} \supseteq A$, so the last ~~exp~~ expression equals A. ∎

<u>Sutherland, Thm. 13.21</u> X, Y compact $\Longrightarrow X \times Y$ compact.

<u>Proof</u> Let $\mathfrak{U}$ = open covering of $X \times Y$ with open sets U_α.

(1) Let $y \in Y$. Then $X \times \{y\} \cong X$ is compact $\Rightarrow$ there is a finite subcovering $U_1, \dots, U_n$ of $X \times \{y\}$. <u>CLAIM</u>: $U_1, \dots, U_n$ covers a thickened slice $X \times W_y$ for some open nbhd W_y of y.

Given $(x,y) \in X \times \{y\}$, Pick U_i such that $(x,y) \in U_i$, then take V_x open in X, V_x' open in Y such that $(x,y) \in V_x \times V_x' \subseteq U_i$. The sets $\{V_x\}$ form an open covering of X. Take a finite sub covering $\{V_{x_j}\}$. If $W_y = \bigcap_j V'_{x_j}$, then $X \times W_y = \bigcup V_{x_j} \times W_y$. But each of these sets is contained in some U_i, so $X \times W_y \subseteq \bigcup_i U_i$.



② The neighborhoods W_y form an open covering of X.

Take a finite subcovering $W_{y_1}, \dots W_{y_s}$

Hence $X \times Y = \bigcup_k X \times W_{y_k}$

For each k we have a finite subcollection $\mathcal{U}_k$ whose union contains $X \times W_k$, so $\bigcup_k \mathcal{U}_k \subseteq \mathcal{U}$ is a finite subcovering of $X \times Y$. ■

<u>Summary</u> $K \subseteq \mathbb{R}^n$ closed $\iff$ closed and bounded.

① K compact $\Rightarrow$ closed, bounded.

② K bounded $\Rightarrow$ $K \subseteq [a_1, b_1] \times \dots \times [a_n, b_n]$

Proof Suppose $v \in K \Rightarrow |v|_2 \leq M$. If $v = (v_1, \dots, v_n)$, then $|v_i| \leq |v|_2 \leq M$ so $K \subseteq [-M, M]^n$.

③ $[-M, M]^n$ is compact.

④ K closed, $K \subseteq [-M, M]^n \Rightarrow K$ is a closed subset of a compact set $\Rightarrow$ K is compact. ■

Further results

Prop 13.15. X compact top. space, $f: X \to Y$ continuous $\Longrightarrow f[X]$ compact
(The continuous image of a compact set is compact.)

Proof. Suppose $\mathcal{U} = \{U_\alpha\}$ is an open covering of $f[X]$. Write $U_\alpha = V_\alpha \cap f[X]$ where V_α is open in Y. Then the sets

$$f^{-1}[V_\alpha] = f^{-1}[V_\alpha \cap f[X]] = f^{-1}[U_\alpha]$$

form an open covering of $X \Longrightarrow$ there is a finite subcovering $f^{-1}[V_{\alpha_1}], \dots, f^{-1}[V_{\alpha_m}]$, or equivalently $f^{-1}[U_{\alpha_1}], \dots, f^{-1}[U_{\alpha_m}]$.
Since $U_\alpha \subseteq f[X]$ we have $f[f^{-1}[U_\alpha]] = U_\alpha$, and therefore $U_{\alpha_1}, \dots, U_{\alpha_m}$ are a finite subcovering of $f[X]$. ∎

Cor. $K \subseteq \mathbb{R}^n$ closed + bdd, $f: K \to \mathbb{R}$ continuous $\Longrightarrow f$ takes a maximum and minimum value. ∎

An inverse function theorem.

Thm 13.26 X compact, Y Hausdorff, $f : X \to Y$ continuous 1-1 onto $\Rightarrow$ f is a homeomorphism.

Proof. Enough to show that E closed in $X \Rightarrow f[E]$ closed in Y.

(This implies that f is open, for V open in X $\Rightarrow X - V$ closed in $X \Rightarrow Y - f[V] = f[Y-V]$ is closed in $Y \Rightarrow f[V]$ open in Y).

since f is 1-1 onto

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But E closed in $X \Rightarrow E$ compact $\Rightarrow$ $f[E]$ compact in Y. Since Y is Hausdorff, this means that $f[E]$ is closed in Y. ∎

See inverse-fcn-thms.pdf for other situations in which f is 1-1 onto continuous $\Rightarrow f$ is a homeomorphism.

Compactness and closed subsets

Thm. A top space X is compact $\iff$:
For each family $\mathfrak{A}$ of closed sets $\{F_\alpha\}$
such that each finite intersection
$F_{\alpha_1} \cap \dots \cap F_{\alpha_n} \neq \emptyset$, we have $\bigcap_{\alpha \in \mathfrak{A}} F_\alpha \neq \emptyset$.

Finite Intersection Property

Cor. Suppose X is compact and we have a sequence $\{F_n\}$ of NONEMPTY closed subsets such that $F_1 \supseteq F_2 \supseteq \dots$, then $\bigcap_n F_n \neq \emptyset$.

Proof of the Corollary. Show $\{F_n\}$ has the finite intersection property, then apply the theorem. But $F_{\alpha_1} \cap \dots \cap F_{\alpha_n} =$
$F_{\mathrm{MAX}\{\alpha_1, \dots, \alpha_n\}}$. ∎

Proof of the theorem.
① Suppose X is compact, and let $\{F_\alpha\}$ have the finite intersection property. If $\bigcap_\alpha F_\alpha = \emptyset$, then $\mathcal{U} = \{X - F_\alpha\}$ is an open covering.

If $X-F_{\alpha_1},\dots,X-F_{\alpha_m}$ is a finite subcovering, then $X = \cup X - F_{\alpha_j} = X - \bigcap_j F_{\alpha_j} \Longrightarrow F_{\alpha_1}\cap\dots\cap F_{\alpha_m} = \phi$. $\boxed{\text{CONTRADICTION}}$

The source of the contradiction was the assumption that $\bigcap_\alpha F_\alpha = \phi$, and hence $\bigcap_\alpha F_\alpha$ must be nonempty. ∎

② Conversely, suppose the condition in the theorem is satisfied. Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of X, let $F_\alpha = X - U_\alpha$. Then

$$\cap F_\alpha = \cap X - U_\alpha = X - \underbrace{\bigcup_\alpha U_\alpha}_{=X} = \phi.$$

The condition in the theorem says this cannot happen if every $F_{\alpha_1}\cap\dots\cap F_{\alpha_m} \neq \phi$, so some finite intersection of this type is nonempty. But if $F_{\alpha_1}\cap\dots\cap F_{\alpha_m} = \phi$, then as above we have $X = U_{\alpha_1}\cup\dots\cup U_{\alpha_m}$, so that $\mathcal{U}$ has a finite subcovering. Hence X is compact. ∎