13a. Compactness and sequences

Chapter 14 of Sutherland covers material involving the behavior of sequences in a compact metric space. The motivation for this topic from first year calculus is less direct than the motivation(s) for Chapters 12 and 13, and the treatment in Sutherland does not say much about this. Some aspects of this are covered in later chapters of Sutherland. We shall concentrate on stating results and applying them to some questions from first year calculus.

Most notable examples. Effectively computable solutions to $f(x) = 0$ for some continuous functions f.

Special case. Standard algorithm for finding $\sqrt{N}$, say N is a positive integer.

<u>Method</u> Rewrite problem into finding a solution to $x = g(x)$ for some fcn. g.

Take $g(x) = \frac{1}{2}\left(x + \frac{N}{x}\right)$, so $g(x) = x$ $\Longleftrightarrow x^2 = N$.

Define a sequence $\{x_n\}$ by starting with $x_0 = N$, $x_{n+1} = g(x_n)$.

<u>CLAIM:</u> For good values of n, $\lim x_n$ exists. Check this works for many choices of N.

If the claim is true, then $L = \lim x_n$ satisfies $L^2 = N$, for

$$g(L) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} x_{n+1}$$ by continuity of g.

so $g(L) = L$, which means that

$$L = \frac{1}{2}\left(L + \frac{N}{L}\right) \text{ or } \frac{1}{2}L = \frac{1}{2}\cdot\frac{N}{L}$$

or $L^2 = N$, or $L = \sqrt{N}$.

Examples to 10 digits

$\begin{pmatrix} 145 = \\ 29 \times 5 \end{pmatrix}$

$N = x_0$	5.0	29.0
x_1	3.0	15.0
x_2	2.333333333	8.466666667
x_3	2.238095238	5.945931759
x_4	2.236068896	5.411608062
x_5	2.236067577	5.385229413
x_6	——— " ———	5.385164808
x_7	——— " ———	5.385164807

SUBSEQUENT APPROXIMATIONS REPEAT x_7

Why does this work so reliably?

$$T(x) = \frac{1}{2}\left(x + \frac{N}{x}\right) \Rightarrow x_n = T^n x_0.$$

<u>Claim</u> There is some c such that $0 < c < 1$ and $|T(u) - T(v)| \le c \cdot |u - v|$ if $\sqrt{N} \le u, v \le N$. Furthermore, T is ~~de~~increasing on the interval $[\sqrt{N}, N]$.

<u>Derivation</u> Suppose $u < v$, so

$$T(u) - T(v) = T'(\alpha) \cdot (u - v) \quad \text{some } \alpha.$$

$T'(x) = \frac{1}{2}\left(1 - \frac{N}{x^2}\right) \Rightarrow$ if $x \geq \sqrt{N}$, then $0 \leq T'(x) \leq \frac{1}{2}$. Hence T is increasing and

$$|T(u) - T(v)| = |T'(\alpha)\cdot(u-v)| = |T'(\alpha)|\cdot|u-v| \leq \frac{1}{2}|u-v|.$$

$\boxed{\text{Can take } c = \frac{1}{2}}$

Since $T(\sqrt{N}) = \sqrt{N}$, it follows that T maps $[\sqrt{N}, N]$ into $[\sqrt{N}, \frac{1}{2}(N+\sqrt{N})]$

Note this is less than N.

can sharpen to $\frac{N+1}{2}$

<u>Claim</u> $\{x_n\} = \{T^n(x_0)\}$ is a Cauchy sequence.

Let $\Delta = d(x_0, x_1)$.

Then $d(x_n, x_{n+1}) = d(T(x_{n-1}), T(x_n)) \leq \frac{1}{2} d(x_{n-1}, x_n)$

$$\ldots \leq \frac{1}{4} d(x_{n-2}, x_{n-1}) \ldots \leq \frac{1}{2^k} d(x_{n-k}, x_{n-k+1})$$

<u>all k</u>

Hence $m > n \Rightarrow$

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots$$

$$+\, d(x_{m-1}, x_m) \leq d(x_n, x_{n+1}) \cdot \underbrace{\left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{\cancel{m-1}}}\right)}_{\text{less than } 2} \cancel{\leq}$$

$m-n-1$

$$\frac{1}{2^n} \cdot d(x_0, x_1) \cdot 2 = \frac{1}{2^{n-1}} d(x_0, x_1).$$

So there is some $M > 0$ such that

$n, m \geq M \Rightarrow d(x_n, x_m) < \varepsilon$. — Choose M

so that $\frac{1}{2^{M-1}} \cdot d(x_0, x_1) < \varepsilon$.

Note that the sequence $\{x_n\}$ lies in the closed interval $[\sqrt{N}, N]$.

Bolzano-Weierstrass Theorem

Every infinite sequence in a closed interval has a convergent subsequence.

Cor. Every Cauchy sequence in a closed interval has a limit.

Idea First find a convergent subsequence of the Cauchy sequence in a closed interval. Then show the limit of the subsequence is the limit of the entire sequence.

Summary of the key steps

① Reformulate problem into solving $g(x)=x$.

② Find a reasonable interval $[a,b]$ such that g maps $[a,b]$ into itself, and

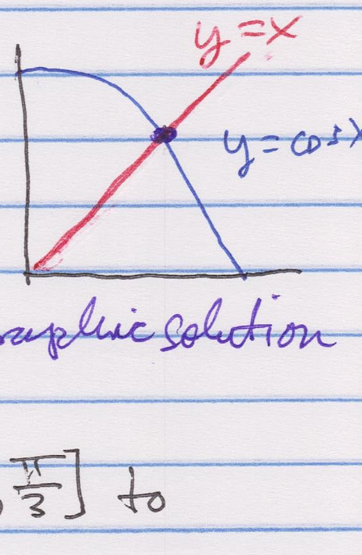
③ g has a continuous derivative such that $|g'(x)| \le c < 1$ on $[a,b]$.

THEN $y = \lim_{n\to} g^n(x)$ satisfies $g(y)=y$.

Sutherland, Thm. 17.26
Prop. 17.9
+ Mean Value Thm.

Another example

Solve $\cos x = x$, $0 \le x \le \frac{\pi}{2}$.



graphic solution

$x=0 \Rightarrow \cos x = 1 > x$.

$x=\frac{\pi}{3} \Rightarrow \cos x = \frac{1}{2} < x$

Also $\frac{\pi}{3} > 1$. Hence cos maps $[0,\frac{\pi}{3}]$ to itself, and $\frac{d}{dx}\cos x = -\sin x \Rightarrow$

$$\left|\frac{d}{dx}\cos x\right| \leq \frac{\sqrt{3}}{2} < 1 \text{ on } \left[0, \frac{\pi}{3}\right]$$

Hence we can solve $\cos x = x$ by taking the limit of the sequence $x_0 = 1$, $x_{n+1} = \cos x_n$. Here are ~~the~~ some successive approximations:

x_5	0.701368724...
x_{10}	0.744237355...
x_{15}	0.738369204...
x_{20}	0.739184400...
x_{25}	0.739071365...
x_{30}	0.739087043...
x_{35}	0.739084868...
x_{40}	0.739085170...
x_{45}	0.739085128...
x_{50}	0.739085134...
x_{55}	0.739085133...

and the approximations after this are the same to the 9th decimal place.

The results in Chapter 14 and the rest of Sutherland provide a framework for solving many vector equations of the form $x = F(x)$, where F is ~~a~~ an n-dim vector valued function defined on an open region in $\mathbb{R}^n$. The final section of Sutherland gives similar applications to satisfying certain functional equations (see pp. 193–198 at the end of Chapter 17).