14. Sequential compactness

An older definition for compact metric spaces, still useful for some purposes.

Sutherland, Thm. 4.19 If $\{a_n\}$ is a bounded infinite sequence in $\mathbb{R}$, then $\{a_n\}$ has a convergent subsequence

Note $\{a_n = n\}$ is unbounded, no such subsequence.

Def. A metric space (X, d) is sequentially compact $\Leftrightarrow$ every infinite sequence has a convergent subsequence.

MAIN THEOREM

Sutherland, Thm. 14.6 (X, d) metric. Then X is compact $\Longleftrightarrow$ X is sequentially compact.

One use: Arzelà - Ascoli Thm. Determine when a subset of $\mathcal{C}$ = cont. fcns. $[0,1] \longrightarrow \mathbb{R}$ has compact closure (uniform norm on $\mathcal{C}$).

References

Rudin, Principles of Mathematical Analysis (3rd Ed), pp. 155-158.

http://en.wikipedia.org/wiki/Arzelà-Ascoli theorem

Def. $(X, \mathcal{T})$ is limit point compact $\Leftrightarrow$ every infinite subset has a limit point.

The usual considerations involving limit pts. and sequences with limits lead to the following

Theorem (X, d) metric space. Then X is limit point compact $\Leftrightarrow$ X is sequentially compact.

Details are worked out in solutions06.pdf (see the solution to Additional Exercise 14.1).

The same exercise yields

Theorem compact $\Rightarrow$ limit point compact ($\Rightarrow$ sequentially compact if (X,d) is a metric space).

CONVERSELY

Theorem (X,d) sequentially compact metric space $\Rightarrow$ X is compact.

$\boxed{\text{Step 1 Under the hypotheses, for each } \varepsilon>0 \text{ there exists a finite collection of sets } N_\varepsilon(x_i) \text{ which cover } X.}$

Proof of Step 1 Suppose for some $\varepsilon > 0$ there is no such finite covering. Define a sequence $\{x_n\}$ in X s.t. $d(x_i, x_j) \geq \varepsilon$ for each $i \neq j$.

Pick any x_1. Given $x_1, \dots, x_{n-1}$ with the property $d(x_i, x_j) \geq \varepsilon$, find x_n by noting that

$$\bigcup_{i=1}^{n-1} N_\varepsilon(x_i) \subsetneq X$$ and picking $x_n \in X -$ union.

Let $S = \{x_1, x_2, \dots\}$, an infinite set.

By sequential compactness, $L(S) \neq \phi$, so let $y \in L(S)$. Take $\{y_m\}$ in S such that the points y_m are distinct and $y_m \to y$.

Then $m \geq N \Rightarrow d(y, y_m) < \frac{\varepsilon}{2}$. Hence

$d(y_m, y_{m+1}) \le d(y, y_m) + d(y, y_{m+1}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Contradiction! — The source is the assumption in the first sentence in the proof, so that assumption is false. Hence the conclusion of the theorem is true. ∎

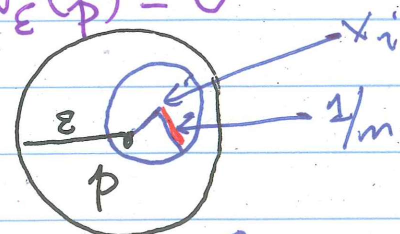
$\boxed{\text{Step 2 Under the hypothesis, the topology on } X \text{ has a countable base.}}$

For $m > 0$ let $\mathcal{U}_m$ be a finite set of nbhds. $N_{1/m}(x_i)$ which cover X, and let

Countable family

$\mathcal{B} = \cup\, \mathcal{U}_m$. To see $\mathcal{B}$ is a base, let U be open in X, let $p \in U$, and let $\varepsilon > 0$ such that $N_\varepsilon(p) \subseteq U$. Choose $\frac{1}{m} < \frac{\varepsilon}{2}$ and x_i such that $p \in N_{1/m}(x_i)$.

CLAIM $N_{1/m}(x_i) \subseteq N_\varepsilon(p) \subseteq U$



Suppose $d(y, x_i) < 1/m$

Then $d(y,p) \le \underbrace{d(y, x_i)}_{< 1/m} + \underbrace{d(x_i, p)}_{< 1/m} < \frac{2}{m} < 2 \cdot \left(\frac{\varepsilon}{2}\right) = \varepsilon$. ∎

Cor. to Step 2 (X, d) compact $\Longrightarrow$ there is a countable base for its topology

NOTE. A result known as the Urysohn metrization theorem implies a converse to Step 2:

If X is compact Hausdorff and the topology has a countable base, then X is homeomorphic to a (compact) metric space.

This follows by combining the discusson on page 195 of Munkres* following the definition with Theorems 32.3 and 34.1 in that book.

Not every compact Hausdorff space is homeomorphic to a metric space. The standard example with these properties is given as Example 105 in Steen and Seebach, Counterexamples in Topology.

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* Munkres, Topology (Second Edition).

Step 3. If $(X, \mathcal{T})$ has a countable base, then every open covering of X has a countable subcovering. (Lindelöf Property)

$\mathcal{U}$ = open covering of X, $\mathcal{B} = \{V_k\}$ a countable base. Let $\{W_j\} = \mathcal{B}_0 \subseteq \mathcal{B}$ be all subsets s.t. $W_j \subseteq U_\alpha$ for some U_α in $\mathcal{U}$. CLAIM $\{W_j\}$ is an open covering of X; this is true since $x \in X \Rightarrow x \in U_\alpha$ some α and some j so that $x \in W_j \subseteq U$. To get a countable subcovering, ~~for each W_j in $\mathcal{B}_0$ pick $j(\alpha)$ so that $W_j \subseteq U$~~ given W_j in $\mathcal{B}_0$, take $\alpha(j)$ so that $W_j \subseteq U_{\alpha(j)}$; then $\{U_{\alpha(j)}\}$ is a countable subcovering.

Conclusion of the proof.

Given an open covering $\mathcal{U}$ of X, take a countable subcovering $\mathcal{U}_0 = \{U_0, U_1, U_2, \dots\}$.

Suppose that $\mathcal{U}_0$ has no finite subcovering. Then each closed set $F_n = X - \bigcup_{i=0}^{n} U_i$ is infinite (otherwise there will be a finite subcovering).

Construct a sequence $\{a_n\}$ recursively so that $a_n \in F_n$ and $a_n \neq a_k$ for $k<n$. Then $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$ with some limit b. Since $\mathcal{U}_0$ covers X, $b \in U_M$ for some M. Pick $\varepsilon > 0$ so that $N_\varepsilon(b) \subseteq U_M$; then for some integer K, $k \geq K$ $\Longrightarrow d(b, a_{n_k}) < \varepsilon$ so that $a_{n_k} \in U_M$.

On the other hand, by construction $a_n \in F_{M+1}$ for $n \geq M+1$, which means $a_n \notin U_M$ for such n. If $n \geq K, M+1$ this yields a contradiction. The source of the contradiction is the assumption that $\mathcal{U}_0$ has no finite subcovering. Hence $\mathcal{U}_0$ (and also $\mathcal{U}$) must have a finite subcovering. ∎

<u>Def.</u> A top. space X is separable if it has a countable dense subset.

<u>Prop.</u> If X has a countable base, then X is separable.

<u>Proof:</u> Let $\mathcal{B} = \{V_0, V_1, \ldots\}$ be a countable base for the topology, and pick $d_i \in V_i$.

<u>CLAIM</u> $D = \{d_0, d_1, \ldots\}$ is dense in X.

Need to show $x \notin D \Rightarrow x \in L(D)$.

Suppose $x \notin D$ and U is an open nbhd of x. Let $x \in V_i \subseteq U$ for a suitable $V_i \in \mathcal{B}$.

a priori we only know $d_i \in V_i$, but we also know $x \notin D$

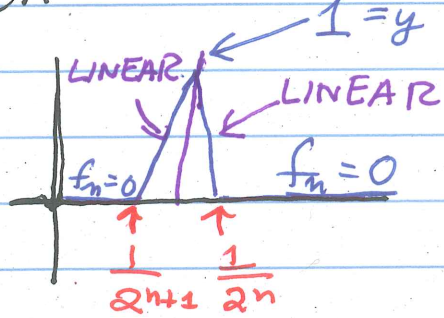
Then $d_i \in (V_i - \{x\}) \cap D \subseteq (U - \{x\}) \cap D$ so the RHS is nonempty, and hence $x \in L(D)$. ■

<u>Corollary</u> A compact metri space is separable.

EXAMPLE $X = \mathcal{C}[0,1]$, continuous fcns. $[0,1] \longrightarrow \mathbb{R}$. Then the closed set

$A = \{ f \in X \mid \|f\| \leq 1 \}$ is not sequentially compact.

f_n graph



Then $m \neq n$

$\Rightarrow \|f_n - f_m\| = 1$

Hence if A is the set of all functions f_n, then $A \underset{\text{ISOMETRIC}}{\cong}$ discrete metric space, so A does not have a convergent subsequence. ■

<u>Sutherland, 17.9 and 17.10.</u> (X, d) compact metric $\Rightarrow$ every Cauchy sequence in X converges.

<u>COROLLARY</u> Every Cauchy sequence in $\mathbb{R}^n$ converges.

<u>PROOF OF COROLLARY</u> Enough to show that $\{x_n\}$ Cauchy $\Rightarrow |x_n| \leq K$ some K.* Let $\varepsilon > 0$ and choose N so that $m, n \geq N \Rightarrow |x_m - x_n| < \varepsilon$. Then $|x_n| \leq$ maximum of $|x_0|, \ldots, |x_{N-1}|, |x_N| + \varepsilon$ because $m \geq N \Rightarrow$

$$|x_m| \leq |x_N| + |x_m - x_N| < |x|_N + \varepsilon. \blacksquare$$

* Since $\{|x| \leq k\}$ is compact in $\mathbb{R}^n$

<u>Proof of the main result.</u>

compact metric $\Rightarrow$ seq compact, so $\{x_n\}$ has a convergent subsequence $\{x_{n(k)}\}$. Let $a = \lim_{k\to\infty} x_{n(k)}$.

Let $\varepsilon > 0$.

CLAIM $a = \lim_{n\to\infty} x_n$.

Choose N_1 so $m, n \geq N_1 \Rightarrow d(x_m, x_n) < \frac{\varepsilon}{2}$.

Choose N_2 so $k \geq N_1 \Rightarrow d(a, x_{n(k)}) < \frac{\varepsilon}{2}$.

Let $N = \max\{N_2, n(N_1)\}$ and choose p so that $n(p) \geq N$, $p \geq N_1$. Then

$$m \geq N \Rightarrow \left.\begin{cases} d(x_m, x_{n(p)}) < \frac{\varepsilon}{2} \quad \leftarrow n(p) \geq N \\ d(a, x_{n(p)}) < \frac{\varepsilon}{2} \quad \leftarrow p \geq N_1 \end{cases}\right\} \text{ so that}$$

$$d(a, x_m) \leq d(a, x_{n(p)}) + d(x_{n(p)}, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and hence $\lim_{n \to \infty} x_n = a$ which is what we wanted. ■