14. Sequential compactness

An older definition for compact metric spaces, still useful for some purposes.

Sutherland, Thm. 4.19 If $\{a_n\}$ is a bounded infinite sequence in $\mathbb{R}$, then $\{a_n\}$ has a convergent subsequence

Note $\{a_n = n\}$ is unbounded, no such subsequence.

Def. A metric space (X, d) is sequentially compact $\Leftrightarrow$ every infinite sequence has a convergent subsequence.

MAIN THEOREM

Sutherland, Thm. 14.6 (X, d) metric. Then X is compact $\Leftrightarrow$ X is sequentially compact.

One use: Arzelà - Ascoli Thm. Determine when a subset of $\mathcal{C}$ = cont. fcns. $[0,1] \longrightarrow \mathbb{R}$ has compact closure (uniform norm on $\mathcal{C}$).

References

Rudin, Principles of Mathematical Analysis (3rd Ed), pp. 155–158.

http://en.wikipedia.org/wiki/Arzelà-Ascoli_theorem

Def. $(X, \mathcal{T})$ is limit point compact $\iff$ every infinite subset has a limit point.

The usual considerations involving limit pts. and sequences with limits lead to the following

Theorem (X, d) metric space. Then X is limit point compact $\iff$ X is sequentially compact.

Details are worked out in solutions06.pdf (see the solution to Additional Exercise 14.1).

The same exercise yields

Theorem compact $\Rightarrow$ limit point compact ($\iff$ sequentially compact if (X, d) is a metric space).