<u>Def.</u> A top. space X is separable if it has a countable dense subset.

<u>Prop.</u> If X has a countable base, then X is separable.

<u>Proof:</u> Let $\mathcal{B} = \{V_0, V_1, \ldots\}$ be a countable base for the topology, and pick $d_i \in V_i$.

<u>CLAIM</u> $D = \{d_0, d_1, \ldots\}$ is dense in X.

Need to show $x \notin D \Rightarrow x \in L(D)$.

Suppose $x \notin D$ and U is an open nbhd of x. Let $x \in V_i \subseteq U$ for a suitable $V_i \in \mathcal{B}$.

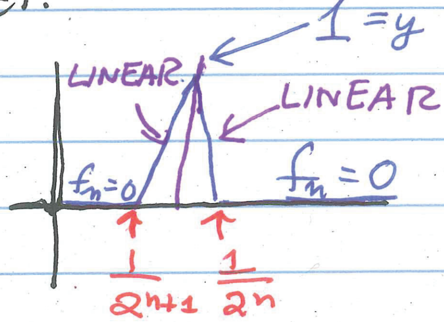
a priori we only know $d_i \in V_i$, but we also know $x \notin D$

Then $d_i \in (V_i - \{x\}) \cap D \subseteq (U - \{x\}) \cap D$ so the RHS is nonempty, and hence $x \in L(D)$. ∎

<u>Corollary</u> A compact metri space is separable.

<u>EXAMPLE</u> $X = \mathcal{C}[0,1]$, continuous fcns. $[0,1] \longrightarrow \mathbb{R}$. Then the closed set $A = \{f \in X \mid \|f\| \leq 1\}$ is not sequentially compact.

f_n graph



Then $m \neq n$ $\Rightarrow \|f_n - f_m\| = 1$

Hence if A is the set of all functions f_n, then $A \underset{\text{ISOMETRIC}}{\cong}$ discrete metric space, so A does not have a convergent subsequence. ■

Sutherland, 17.9 and 17.10. (X,d) compact metric $\Rightarrow$ every Cauchy sequence in X converges.

COROLLARY Every Cauchy sequence in $\mathbb{R}^n$ converges.

PROOF OF COROLLARY Enough to show that $\{x_n\}$ Cauchy $\Rightarrow |x_n| \le K$ some K.* Let $\varepsilon > 0$ and choose N so that $m,n \geq N \Rightarrow |x_m - x_n| < \varepsilon$. Then $|x_n| \le$ maximum of $|x_0|, \dots, |x_{N-1}|, |x_N| + \varepsilon$ because $m \geq N \Rightarrow$

$$|x_m| \le |x_N| + |x_m - x_N| < |x|_N + \varepsilon. \blacksquare$$

* Since $\{|x| \le K\}$ is compact in $\mathbb{R}^n$

Proof of the main result.

compact metric $\Rightarrow$ seq compact, so $\{x_n\}$ has a convergent subsequence $\{x_{n(k)}\}$. Let $a = \lim_{k\to\infty} x_{n(k)}$.

Let $\varepsilon > 0$.

CLAIM $a = \lim_{n\to\infty} x_n$.

Choose N_1 so $m, n \geq N_1 \Rightarrow d(x_m, x_n) < \frac{\varepsilon}{2}$.

Choose N_2 so $k \geq N_1 \Rightarrow d(a, x_{n(k)}) < \frac{\varepsilon}{2}$.

Let $N = \max\{N_2, n(N_1)\}$ and choose p so that $n(p) \geq N$, $p \geq N_1$. Then

$$m \geq N \Rightarrow \left\{ \begin{array}{l} d(x_m, \underset{\uparrow\; n(p) \geq N}{x_{n(p)}}) < \frac{\varepsilon}{2} \\[2ex] d(a, \underset{\uparrow\; p \geq N_1}{x_{n(p)}}) < \frac{\varepsilon}{2} \end{array} \right\} \text{ so that}$$

$$d(a, x_m) \leq d(a, x_{n(p)}) + d(x_{n(p)}, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and hence $\lim_{n \to \infty} x_m = a$ which is what we wanted. ■