Introduction to Metric and Topological Spaces
by Wilson Sutherland
Unofficial Solutions Manual

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Dedicated to my Parents

Preface

This is an ongoing Solutions Manual for *Introduction to Metric and Topological Spaces* by Wilson Sutherland [1]. The main reason for taking up such a project is to have an electronic backup of my own handwritten solutions.

Mathematics cannot be done without actually *doing* it. However at the undergraduate level many students are put off attempting problems unless they have access to written solutions. Thus I am making my work publicly available in the hope that it will encourage undergraduates (or even dedicated high school students) to attempt the exercises and gain confidence in their own problem-solving ability.

I am aware that questions from textbooks are often set as assessed homework for students. Thus in making available these solutions there arises the danger of plagiarism. In order to address this issue I have attempted to write the solutions in a manner which conveys the general idea, but leaves it to the reader to fill in the details.

At the time of writing this work is far from complete. While I will do my best to add additional solutions whenever possible, I can not guarantee that any one solution will be available at a given time. Updates will be made whenever I am free to do so.

I should point out that my solutions are not the only ways to tackle the questions. It is possible that many 'better' solutions exist for any given problem. Additionally my work has not been peer reviewed, so it is not guaranteed to be free of errors. Anyone using these solutions does so at their own risk.

I also wish to emphasize that this is an *unofficial* work, in that it has nothing to do with the original author or publisher. However, in respect of their copyright, I have chosen to omit statements of all the questions. Indeed it should be quite impossible for one to read this work without having a copy of the book [1] present.

I hope that the reader will find this work useful and wish him the best of luck in his Mathematical studies.

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The end of a solution is indicated by ■. Any reference such as 'Proposition 2.3.13', 'Definition 3.8.1', 'Question 10.3.16' refers to the relevant numbered item in Sutherland's book [1]. This work has been prepared using LaTeX.

The latest version of this file can be found at: `http://akhtarmath.wordpress.com/`

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Chapter 1

Review of some real analysis

1.5.1) We assume that A and B are nonempty. Since B is bounded above, $\sup B$ exists and is finite. Since $A \subseteq B\colon a \in A \Rightarrow a \in B \Rightarrow a \leq \sup B$. So A is bounded above and $\sup B$ is an upper bound for A. Therefore, $\sup A$ exists and $\sup A \leq \sup B$. ■

1.5.2) We assume that A and B are nonempty. Since $A, B \subset \mathbb{R}$ are bounded above, $\sup A$ and $\sup B$ both exist and are finite. In particular $\mathrm{Max}\{\sup A, \sup B\}$ exists and is finite. Now if $x \in A \cup B$ then either $x \in A \Rightarrow x \leq \sup A \leq \mathrm{Max}\{\sup A, \sup B\}$ or $x \in B \Rightarrow x \leq \sup B \leq \mathrm{Max}\{\sup A, \sup B\}$. In either case we see that $A \cup B$ is bounded above and that $\mathrm{Max}\{\sup A, \sup B\}$ is an upper bound for $A \cup B$. Let U be any upper bound for $A \cup B$. Then $\sup A \cup B \leq U$. Since $A \subseteq A \cup B \subseteq \mathbb{R}$ it follows from Question 1.5.1 that $\sup A \leq \sup A \cup B \leq U$. Similarly $\sup B \leq U$. Thus $\mathrm{Max}\{\sup A, \sup B\} \leq U$. We conclude that $\mathrm{Max}\{\sup A, \sup B\} = \sup A \cup B$. ■

Chapter 2

Continuity generalized: metric spaces

2.6.1) If there exists $p \in B_r(x) \cap B_r(y)$ then $d(p, x) < r$ and $d(p, y) < r$ so that $2r = d(x, y) \leq d(p, x) + d(p, y) < 2r$, which is a contradiction. ■

2.6.2) $|d(x, z) - d(y, z)| \leq d(x, y) \iff -d(x, y) \leq d(x, z) - d(y, z)$ and $d(x, z) - d(y, z) \leq d(x, y) \iff d(y, z) \leq d(x, y) + d(x, z)$ and $d(x, z) \leq d(x, y) + d(y, z)$. The last two inequalities are true by the triangle inequality so the result follows. ■

2.6.3) Proceed exactly as in Question 2.6.2. $d(x, z) + d(y, t) \geq |d(x, y) - d(z, t)| \iff d(x, z) + d(y, t) \geq d(x, y) - d(z, t)$ and $-d(x, z) - d(y, t) \leq d(x, y) - d(z, t) \iff d(x, z) + d(y, t) + d(z, t) \geq d(x, y)$ and $d(z, t) \leq d(x, y) + d(x, z) + d(y, t)$. The last two inequalities are true by repeated application of the triangle inequality, so the result follows. ■

2.6.4) Let $X := \{d(s, t) \mid s, t \in C\}$ and $Y := \{d(p, q) \mid p, q \in B\}$. Then $\operatorname{diam} B = \sup Y$ and $\operatorname{diam} C = \sup X$. Also $C \subseteq B$, which implies that $X \subseteq Y$ ($\subseteq \mathbb{R}$). A subset of a metric space is bounded if and only if its diameter is finite. Since B is bounded, $\operatorname{diam} B = \sup Y$ is finite. In particular, Y is bounded above and $X \subseteq Y \subseteq \mathbb{R}$, so it follows[1] that $\sup X \leq \sup Y < \infty$. The fact that $\operatorname{diam} C = \sup X < \infty$ tells us that C is bounded, and the result $\sup X \leq \sup Y$ is equivalent to saying that $\operatorname{diam} C \leq \operatorname{diam} B$. ■

2.6.5) Let $A := \{d(x, y) \mid x, y \in B \cup C\}$ so that $\sup A = \operatorname{diam}(B \cup C)$. Note that $A = S \cup T \cup V$, where $S := \{d(m, n) \mid m, n \in B\}$, $T := \{d(p, q) \mid p, q \in C\}$, $V := \{d(h, k) \mid h \in B \backslash C, k \in C \backslash B\}$. Furthermore, $\operatorname{diam} B = \sup S$, $\operatorname{diam} C = \sup T$ and $\operatorname{diam} B, \operatorname{diam} C \geq 0$. Select any $a \in A$. There are three possible cases. If $a \in S$ then $a \leq \sup S = \operatorname{diam} B \leq \operatorname{diam} B + \operatorname{diam} C$. Similarly, if $a \in T$ then $a \leq \operatorname{diam} B + \operatorname{diam} C$. Finally, if $a \in V$ then $a = d(h, k)$ for some $h \in B \backslash C$ and $k \in C \backslash B$. Choose[2] any $x \in B \cap C$. Then $a = d(h, k) \leq d(h, x) + d(x, k)$ with $d(h, x) \in S$ and $d(x, k) \in T$. It follows once again that $a \leq \sup S + \sup T = \operatorname{diam} B + \operatorname{diam} C$. In all cases we see that $\operatorname{diam} B + \operatorname{diam} C$ is an upper bound for A so that $\operatorname{diam}(B \cup C) = \sup A \leq \operatorname{diam} B + \operatorname{diam} C$ as required. ■

2.6.10) Let S be a subset of a metric space. Any open ball in a metric space is an open set in that space. The union of any family of open sets in a metric space is also open. Thus if S is a union of open balls then S must be open. Conversely suppose that S is open. Then for any $x \in S$ there exists a real number $r(x) > 0$ such that $B_{r(x)}(x) \subseteq S$. In fact $S = \bigcup_{x \in S} B_{r(x)}(x)$ i.e. S is a union of open balls. ■

[1] Using Question 1.5.1

[2] We can do this because $B \cap C \neq \varnothing$.

2.6.12) No. To see this let (M, d) be any metric space with at least two distinct points which we will call x and y. Since $x \neq y$, $d(x, y) \neq 0$. Say $d(x, y) = 2r > 0$. We claim that neither of the open sets $B_r(x), B_r(y) \subseteq M$ are contained in $\{\varnothing, M\}$. The fact that $x \in B_r(x)$ and $y \in B_r(y)$ guarantees that $B_r(x), B_r(y) \notin \{\varnothing\}$. Also[3] $B_r(x) \cap B_r(y) = \varnothing$ so that $B_r(x), B_r(y) \notin \{M\}$. Therefore if M is any metric space containing at least two distinct points then M contains at least two open sets other than $\varnothing$ and M. ■

2.6.13) Set $\varepsilon = f(a) > 0$. Since $f : M \to \mathbb{R}$ is continuous at $a \in M$, there exists a $\delta > 0$ such that $x \in B_\delta(a) \Rightarrow |f(x) - f(a)| < \varepsilon = f(a) \Rightarrow f(x) > 0$. ■

[3] Using Question 2.6.1

Chapter 4

The Hausdorff condition

4.3.1) As in Example 3.1.7 [1, p.47], consider $\mathbb{R}$ with the Zariski topology. Suppose the topology is Hausdorff and select $x, y \in \mathbb{R}$ with $x \neq y$. Then there exist Zariski open sets U, V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$. But then $U^c \cup V^c = \mathbb{R}$. This is a contradiction since both U^c and V^c are finite while $\mathbb{R}$ is infinite. So the Zariski Topology is not Hausdorff. ■

4.3.4) (a) Let S be the intersection of all the open subsets of T that contain x. Then $\{x\} \subseteq S$. Suppose $S \nsubseteq \{x\}$. Then there exists $y \in S$ is such that $y \notin \{x\}$ i.e. $y \neq x$. Since T is Hausdorff, there exist open sets A, B such that $x \in A$, $y \in B$ and $A \cap B = \varnothing$. In particular, $y \notin A$. But $y \in S \subseteq A$. Contradiction. So $S \subseteq \{x\}$. We conclude that $\{x\} = S$.
(b) We need to give an example of a non-Hausdorff space W in which the following holds true : For any $x \in W$, the intersection of all open subsets of W that contain x is equal to $\{x\}$. Consider $\mathbb{R}$ with the Zariski topology. This is a non-Hausdorff space[1]. Now select any $x \in \mathbb{R}$. Let S be the intersection of all the Zariski open subsets of $\mathbb{R}$ that contain x. Then $\{x\} \subseteq S$. Suppose that $S \nsubseteq \{x\}$. Then there exists $y \in S$ such that $y \notin \{x\}$. Now $y \neq x$ belongs to every Zariski open subset of $\mathbb{R}$ that contains x. It follows that $y \in \mathbb{R}\backslash\{y\}$. Contradiction. Therefore, $S \subseteq \{x\}$. It follows that $\{x\} = S$. ■

[1] By Question 4.3.1

Chapter 5

Compact spaces

5.10.1) Fix any $n \in \mathbb{N}$ and let $X := \{x_1, \ldots, x_n\}$ be a topological space. For some indexing set I let $\{C_i\}_{i \in I}$ be an open cover of X. If $k \in \{1, \ldots, n\}$ then $x_k \in X$ and we know that $X = \bigcup_{i \in I} C_i$. So there exists $C_{i_k} \in \{C_i\}_{i \in I}$ such that $x_k \in C_{i_k}$. It follows that $\{C_{i_1}, \ldots, C_{i_n}\}$ is a finite subcover of $\{C_i\}_{i \in I}$. Therefore X is compact. ■

5.10.2) Let X be a discrete space that is compact. Suppose that X is infinite. Then $\{\{p\}\}_{p \in X}$ is an open cover of X with no finite subcover. This contradicts the assumption that X is compact. Therefore X must be finite. ■

5.10.3) Let C be a collection of open sets in T that covers $H \cup K$ so that $H \cup K \subseteq \bigcup_{V \in C} V$. Since $H \subseteq H \cup K$, C is also an open cover for H. Since H is compact, there exists a finite sub-collection $\{H_1, \ldots, H_n\} \subseteq C$ such that $H \subseteq \bigcup_{i=1}^{n} H_i$. Similarly, there exists a finite sub-collection $\{K_1, \ldots, K_l\} \subseteq C$ such that $K \subseteq \bigcup_{j=1}^{l} K_j$. Then $\{H_1, \ldots, H_n, K_1, \ldots, K_l\} \subseteq C$ is a finite subcover of C for $H \cup K$. ■

5.10.11) A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded[1]. If $A \subseteq \mathbb{R}$ is not bounded then the continuous function $f : A \to \mathbb{R}$ such that $f(a) = a$ is not bounded. On the other hand, if A is not closed then $A \neq Cl(A)$. So there must exist at least one limit point l of A such that $l \notin A$. Define $g : A \to \mathbb{R}$ by $g(a) = 1/(l - a)$. Then the function g is continuous but not unbounded. ■

5.10.14) Consider the function $g : T \to \mathbb{R}$ defined by $g(x) = d(f(x), x)$. First we we will show that g is continuous. Select any $a \in T$ and fix $E > 0$. We require a $D > 0$ such that $d(x, a) < D \Rightarrow |g(x) - g(a)| = |d(f(x), x) - d(f(a), a)| < E$. Now[2] $|d(f(x), x) - d(f(a), a)| \leq d(f(x), f(a)) + d(x, a)$. Furthermore, since $f : T \to T$ is continuous at $a \in T$, there exists $\delta > 0$ such that $d(y, a) < \delta \Rightarrow d(f(y), f(a)) < E/2$. Set $D = \text{Min}\{\delta, E/2\}$. Then $d(x, a) < D \Rightarrow |g(x) - g(a)| = |d(f(x), x) - d(f(a), a)| \leq d(f(x), f(a)) + d(x, a) < (E/2) + (E/2) = E$. This establishes the continuity of g. Next observe that $g(T) = \{d(f(x), x) \,|\, x \in T\}$. Since $g : T \to \mathbb{R}$ is continuous, and T is compact, $g(T)$ is bounded and g attains its bounds on T. In particular, there exists $t \in T$ such that $g(t) = \inf g(T)$. Additionally, since $f(t) \neq t$, $g(t) = d(f(t), t) > 0$. Set $\varepsilon = g(t) > 0$. Then $\varepsilon = \inf g(T) \leq g(x) = d(f(x), x)$ for all $x \in T$. ■

5.10.18) Let d_i denote he metric on M_i $(i = 1, 2, 3)$. Fix $\varepsilon > 0$. There exists a $\delta_g > 0$ such that for all $a, b \in M_2$: $d_2(a, b) < \delta_g \Rightarrow d_3(g(a), g(b)) < \varepsilon$. Also there exists $\delta_f > 0$ such that for all $x, y \in M_1$: $d_1(x, y) < \delta_f \Rightarrow d_2(f(x), f(y)) < \delta_g$. Set $\delta = \delta_f$. Then for all $x, y \in M_1$: $d_1(x, y) < \delta = \delta_f \Rightarrow d_2(f(x), f(y)) < \delta_g \Rightarrow d_3((g \circ f)(x), (g \circ f)(y)) < \varepsilon$.

[1] A special case of the corresponding statement for $\mathbb{R}^n$.

[2] Using Question 2.6.3

Therefore, $g \circ f : M_1 \to M_3$ is uniformly continuous. ∎

5.10.19) Let $(M_1, d_1), (M_2, d_2)$ be metric spaces and $f : M_1 \to M_2$ be a given function. f is not uniformly continuous on M_1 if and only if there exists an $\varepsilon > 0$ such that for all $\delta > 0$, there exist $x, y \in M_1$ satisfying both $d_1(x, y) < \delta$ and $d_2(f(x), f(y)) \geq \varepsilon$. In order to show that $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$, set $\varepsilon = 1$ and choose any $\delta > 0$. Let m, n be positive integers such that $x := 1/m < \delta/2$ and $y := 1/n < \delta/2$. Then $x, y \in (0, 1)$ are such that: $|x - y| \leq |x| + |y| = (1/m) + (1/n) < (\delta/2) + (\delta/2) = \delta$ and $|(1/x) - (1/y)| = |m - n| \geq 1 = \varepsilon$. So f is not uniformly continuous on $(0, 1)$. ∎

Chapter 7

Compactness again: convergence in metric spaces

7.3.1) Let (a_n) be a convergent sequence in the metric space (X, d) with limit[1] $l \in X$. Fix $\varepsilon > 0$. There exists $N(\varepsilon) \in \mathbb{N}$ such that $n \geq N(\varepsilon) \Rightarrow d(a_n, l) < \frac{\varepsilon}{2}$. Therefore $m, n \geq N(\varepsilon) \Rightarrow d(a_m, a_n) \leq d(a_m, l) + d(a_n, l) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ which implies that (a_n) is a Cauchy sequence. ■

7.3.2) Let (z_n) be a Cauchy sequence in the metric space (X, d). Fix $\varepsilon = 1 > 0$. Then there exists N such that $m, n \geq N \Rightarrow d(z_m, z_n) < 1$. Fix $m = N+1 > N$. Then $n \geq N \Rightarrow d(z_1, z_n) \leq d(z_1, z_m) + d(z_m, z_n) < d(z_1, z_m) + 1$. Set $M = \max\{d(z_1, z_1), d(z_1, z_2), \ldots, d(z_1, z_{N-1})\}$. Then $d(z_1, z_n) \leq \max\{M, d(z_1, z_m) + 1\}$ for all n. So the Cauchy sequence (z_n) is bounded. ■

[1] Since every metric space is Hausdorff, the limit is unique.

Bibliography

[1] Sutherland, W. *Introduction to Metric and Topological Spaces*, 1975. Oxford Science Publications.