SOLUTIONS TO EXERCISES FOR

MATHEMATICS 144 — Part 1

Fall 2006

(*Complete through Section* **II.1**)

NOTE. Strictly speaking, the material in the exercises called *Questions to consider* will not be covered on examinations, and similarly for the two groups of exercises for Section II.0 (*Problems from Rosn* and *Additional exercises*), but an understanding of proofs at the level of the exercises for Section II.0 will at least be useful, and it would also be good to understand the responses given here to the questions for Section **II.1**.

I. General considerations

I.1 : Overview of the course

Questions to consider

GENERAL REMARK. There are many possible correct answers to the questions for this section, and the ones given below are just typical examples.

1. No one wants a building to have serious malfunctions when subjected to everyday stresses and strains. Constant repairs of the supporting structure make it impossible to do the work that one needs to do inside the building. Similarly, if mathematical foundations are not constructed carefully, it is far more likely that problems with them will develop all the time and interfere with the attempts to use mathematics for understanding concepts and problems.∎

2. Careful preparation and designing often save time and energy in the long run, more than repaying the initial investment needed to set them up and allowing one to do things that would otherwise be very difficult or impossible. The same principle holds for mathematics. Good formulations can make it much easier to understand a problem and solve it efficiently or successfully.∎

3. If one dwells to much on small details for their own sake, this can disrupt efforts to understand the original problem.∎

4. Crash tests for automobiles are one example as are similar destruction tests to determine the strength of objects like boxes or other containers. Heat tests to determine safe usages for consumer or industrial products are another example.∎

I.2 : Historical background and motivation

Questions to consider

1. The integral formula is just an extremely precise and convenient approximation to the physical center of mass.

2. The argument assumes that the angle sum for all triangles is the same number S, and the possibility of different sums for different triangles is overlooked.■

3. The equation of the line has the form $y - \frac{1}{2} = m(x - \frac{1}{2})$. One needs to show that each of these lines, and also the vertical line $x = \frac{1}{2}$ contains either a point with coordinates $(0, t)$ such that $0 \leq t \leq 1$ or a point with coordinates $(t, 0)$ such that $0 \leq t \leq 1$. This can be done using a case by case argument for different choices of m and also for the vertical line. We shall merely list the different cases and describe what happens. Drawing pictures for each case is strongly recommended, and using the pictures one can then proceed to solve equations and prove that the coordinates of the solution have the indicated form. Note that the slope m cannot be equal to -1 because that is the slope of the line joining $(0, 1)$ and $(1, 0)$.

$m = 1$. — This line also goes through the origin, which lies on the triangle.

$m > 1$. — These lines also goes through a point between $(0, 0)$ and $(1, 0)$ of the form $(t, 0)$ where t lies between 0 and $\frac{1}{2}$.

$m = \infty$ (vertical line). — This lines also goes through the point $(\frac{1}{2}, 0)$.

$m < -1$. — These lines also goes through a point of the form $(t, 0)$ where t lies between $\frac{1}{2}$ and 1.

$-1 < m < 1$. — These lines also goes through a point of the form $(0, t)$ where t lies between 0 and 1.■

4. There are four possibilities for the distribution of cards. Let A denote the number of players receiving a picture card from the first deck and a number card from the second, let B be the number of players receiving two picture cares, let C be the number of players receiving a number card from the first deck and a picture card from the second, and let D be the number of players receiving two number cards.

We then have

$$A + B + C + D = 52$$

which is the total number of players and cards in each deck. Since there are 12 picture cards and 40 number cards in each deck we also have

$$A + B = B + C = 12 \qquad C + D = A + D = 40$$

and since all numbers are nonnegative it follows that $A \leq 12$ and $D \leq 40$. Thus we have

$$D = 40 - A \geq 40 - 12 = 28$$

which is larger than half the number of players.

5. Let x_n denote the n^{th} partial sum of the series. Then

$$s_{3n} = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{4n-2} - \frac{1}{4n}\right)$$

so that $\lim_{n\to\infty} s_{3n} = \frac{1}{2}\ln 2$. Also, we have

$$s_{3n+1} = s_{3n} + \frac{1}{2n+1} \qquad s_{3n+2} = s_{3n} + \frac{1}{4n+2}$$

so that $\lim_{n\to\infty} s_{3n+1} = \lim_{n\to\infty} s_{3n} = \lim_{n\to\infty} s_{3n+2}$. We need to piece these together to prove that $\lim_{n\to\infty} s_n = \frac{1}{2}\ln 2$.

We now know the limits of the three sequences with terms s_{3n}, s_{3n+1}, s_{3n+2} exist and have the same value L. Therefore, for each $\varepsilon > 0$ we can find some positive integer M such that $m \geq M$ implies $|y_m - L| < \varepsilon$ for each of the sequences we have considered. Now suppose that $n \geq N$. Write $n = 3m + r$ where m is a nonnegative integer and r is one of 0, 1, 2. If $M \geq 3M + 3$, then $m \geq M$ and hence we have $|s_n - L| = |s_{3m+r} - L| < \varepsilon$. By the previous paragraph we know that $L = \frac{1}{2}\ln 2$, so this completes the proof.■

6. If we substitute $x = \pi/2$ into the series we obtain

$$-\frac{4}{\pi}\cdot(1 \; - \; 3 \; + \; 5 \; \cdots)$$

which is a divergent series and as such does not converge to zero.■

I.3 : Selected problems

Questions to consider

1. It is not possible to impose such conditions because one always can put all the objects into the same box. However, if we limit the number of objects that can be put into a given box, then in some cases we can conclude that some box must contain at least two objects. In the suggested example, suppose that box k contains a_k objects, where we insist that $a_k < 3$ for all k. Then we have $\sum_k a_k = 2n$; if each a_k is less than 2, then since there are n terms it follows that the sum is at most n. Note that this sort of argument actually proves a little more; namely, every box must contain exactly two elements under the extra condition.■

2. This is just an algebraic exercise involving geometric series. The right hand side is equal to

$$\frac{2}{8}\left(1+\frac{1}{8^2}\;\cdots\right) \;+\; \frac{5}{8^2}\left(1+\frac{1}{8^2}\;\cdots\right)$$

so all one needs to do is note that the sum of the geometric series which appears twice in this expression is equal to 64/63 and simplify the resulting expression to check it is equal to 1/3.■

3. The simplest way to get some insight into y^3 is to cube the equation $x = y - 1$. This yields

$$2 \;=\; x^3 \;=\; y^3 \;-\text{`}3y^2 \;+\; 3y \;-\; 1$$

which simplifies to $y^3 = 3y^2 - 3y + 3$. If we subtract the expression on the right hand side from both sides of the equation, we obtain a nontrivial cubic polynomial which has y as one of its roots.■

II. Basic concepts

II.0 : Topics from Logic

Problems from Rosen

Some of these are taken fairly directly from two solutions manuals that have been published to be used with that text; the Student Solutions Guide (ISBN 0–07–247477–7) contains solutions for the odd-numbered exercises, and the Instructor's Resource Guide (ISBN 0–07–247480–7) contains solutions for the even-numbered exercises.

1. (*Sketch*) In order to show that one is equivalent to the other, it is necessary to look at all possible cases (p and q are both true, p is true but q is false, p is false but q is true, p and q are both false) and check that the truth values of the two compound statements are the same in all these cases. Direct checking shows that each of the latter will be true in the second and third cases but false in the others.

2. (*Sketch*) Apply the same idea as in the previous exercise to show the equivalence. Verifying that all three operations can be expressed using Sheffer's stroke starts by observing that NOT is so expressible. At the next step we see that AND is expressible in terms of NOT and Sheffer's stroke, so by the first step it can be expressed entirely in terms of the latter. Finally, at the third step we see that OR can be expressed in terms of NOT and AND; since each of the latter can be expressed entirely in terms of Sheffer's stroke, it follows that OR can also be so expressed.

3. An implication is true if the hypothesis is false, so it is easy for the second compound statement to be true if we take $P(x)$ to be any statement that is not always true. For examples, suppose let $P(x)$ denote, "x is an even number." If we now take (x) to be the statement, "x is divisible by 4," then the first compound statement will be false, but the first will be true.

4. Both are true precisely when at least one of the statements $P(x)$ and $Q(x)$ is true for at least one admissible choice of x.

5. It suffices to find a counterexample. Let $P(x)$ be the statement that x is an even number, and let $Q(x)$ be the statement that x is an odd number. Then the first compound statement is true (every number is even or odd) but the second (all numbers are even or all numbers are odd) is false.

6. Take $P(x)$ and $Q(x)$ as in the previous exercise. Then the first statement (there is a number that is both even and odd) is false, but the second (there is an even number and there is an odd number) is true.

7. A less formal way of expressing $P(x, y)$ is to say that student x has taken class y. In these terms, here are the everyday versions of the statements in the exercises:

(a) There is some student who has taken some class.
(b) There is some student who has taken all the classes.
(c) Every student has taken some class.
(d) There is a class that every student has taken.
(e) Every class has been taken by some student.
(f) Every student has taken every class.

8. We shall do them in order.

(a) There is some x such that $x + y = y$ for all y.
(b) For all numbers x and y, if x is nonnegative and y is negative, then $x - y$ is nonnegative.
(c) For all numbers x and y, the product xy is nonzero if and only if both x and y are nonzero.
(d) There are numbers x and y such that $x^2 > y$ and $x < y$.
(e) For all numbers x and y, there is a number z such that $x + y = z$.
(f) For all numbers x and y, if x and y are negative then their product xy is positive.

9. In fact, the first number is not a perfect square, for if it could be written as n^2 for some positive integer n then the rational number $n/10^{250}$ would be the square root of 2. Since $\sqrt{2}$ is irrational, this yields a proof by contradiction. This proof is constructive because we explicitly describe a number from the possibilities in the theorem which is not a perfect square.

10. It suffices to observe that $9 = 3^2$ and $8 = 2^3$ satisfy the given condition.

Note. Recently P. Mihăilescu proved a conjecture made in the 19^{th} century by E. Catalan (1814–1894); namely, that this is the **only** pair of consecutive positive integers which can be expressed as a^x and b^y, where a, b, x, y are all positivie integers and the exponents x, y are greater than 1 (of course, the answer to the problem is no for trivial reaons if we allow either exponent to equal 1). The proof is at a very advanced level, but for the sake of completeness here is a reference: P. Mihăilescu, *Primary cyclotomic units and a proof of Catalan's Conjecture*, [Crelle] Journal für die reine und angewandte Mathematik **572** (2004), 167–195.

11. Each of the three numbers is either nonnegative or nonpositive, so at least two of them, say m and n, are of the same type (positive or negative). But this means their product is nonnegative. This proof is nonconstructive because we are only saying that one of the products is nonnegative and do not specify which one(s) might satisfy the condition. (Actually all three numbers turn out to be positive and hence all the pairwise products are too.)

12. To prove existence, suppose that n is odd, and write it as $2k + 1$ for some other integer k. Then simple algebra shows that n is equal to $(k - 2) + (k + 3)$. — To prove uniqueness, suppose that m is any integer such that $n = (m - 2) + (m + 3)$. If we simplify and use the previous expression for n, we obtain the equation $2k + 1 = 2m + 1$, and we can now use elementary algebra to conclude that $m = k$.

13. A less formal way of stating $P(x, y)$ is to say that the number m divides the number n (evenly, with no remainder). In these terms, here are the answers with explanations in some cases:
(a) FALSE. Certainly 4 does not divide 5.
(b) TRUE.
(c) FALSE. Some numbers do not divide others.
(d) TRUE. The number 1 evenly divides all numbers.
(e) FALSE. The first part gives a counterexample.
(f) TRUE. This follows from the comment in the third part.

14. We shall prove the contrapositive: *If x is rational, then so is x^3.* — The product of two rational numbers is rational, so x^2 is rational, and hence $x^3 = x^2 \cdot x$ must also be rational.

15. Consider the Pythagorean triples in the hint. We have $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$. Suppose we multiply the first equation by 13^2 and the second by 5^2. Then after doing some algebra we find that

$$39^2 + 52^2 = 65^2 = 25^2 + 60^2$$

and hence 65^2 is written as a sum of two squares in two different ways.

In the preceding example, the sum of the two squares is itself a perfect squares; if one is willing to take sums of two squares that are not necessarily perfect squares, then there are numerous smaller examples such as $5^2 + 5^2 = 50 = 7^2 + 1$ or $4^2 + 7^2 = 65 = 8^2 + 1^2$ or $6^2 + 7^2 = 85 = 9^2 + 2^2$.

16. The first few cubes are 1, 8 and 27; if we want to find a number that cannot be written as a sum of eight cubes, we might look for a number that is 7 more than some small multiple of 8. In fact, we cannot write 23 in the prescribed manner. Certainly this is impossible if we use all 1's or one 8, and if we use two 8's we also need seven 1's, and hence we need at least nine cubes to write 23. In fact, this turns out to be the smallest possible counterexample.

Note. A proof of Lagrange's theorem on expressing a positive integer as a sum of four (or fewer) perfect squares is given in Section 7.4 of the following book: I. N. Herstein, *Topics in Algebra* (2nd Ed.), Wiley, New York, 1975, ISBN 0– 571–01090–1. — The proof is nominally at the advanced undergraduate level, but it might be more accurate to place it at the beginning graduate level.

17. In fact, 7 is not a sum of two squares and the cube of a nonnegative integer. Any such expression for a number less than 8 must be a sum of 4's and 1's, and at least four such numbers are needed to obtain a sum of 7. Once again, this is the smallest possible counterexample.

18. There are several sequences of steps that will achieve the stated goal, and we shall given the one in the supplement to Rosen. At each stage, let (a, b, c) denote the contents of the jugs holding 8, 5 and 3 gallons respectively. Then at the initial stage we have $(8, 0, 0)$. If we fill the 5 gallon jug using the 8 gallon jug, we get the configuration $(3, 5, 0)$. Now fill the 3 gallon jug using the 5 gallon jug to get the distribution $(3, 2, 3)$. Pour the contents of the 3 gallon jug back into the 8 gallon jug so that we have $(6, 2, 0)$, and pour the contents of the 5 gallon jug into the 3 gallon jug so that we have $(6, 0, 2)$. Next, fill the 5 gallon jug using the 8 gallon jug to obtain a distribution of $(1, 5, 2)$. Finally, top off the 3 gallon jug using the 5 gallon jug; this leaves us with $(1, 4, 3)$ and hence the 5 gallon jug now has 4 gallons of water and thus we have measured out 4 gallons of water as asked for in the problem.

Additional exercises to work

1. Suppose that P is the statement that x is the real number zero, Q is the statement that x is the real number one, and R is the statement that x is a real number. Then both $P \vee R$ and $Q \vee R$ are equivalent to R, but certainly P is not logically equivalent to Q.

Similarly, suppose P is the statement that the integer x is a perfect square, Q is the statement that the integer x is a perfect cube, and R is the statement that the intege x is a sixth power. Then both $P \wedge R$ and $Q \wedge R$ are logically equivalent to R, but P and Q are not logically equivalent because there are integers that are perfect squares but not perfect cubes and vice versa.∎

2. The statement $\exists x \forall y Q(x,y)$ asserts there is an odd integer x such that for all odd integers x the number y^x is a perfect square. This is false. If x is odd then 3^x is never a perfect square. The statement $\forall y \exists x Q(x,y)$ asserts for every odd integer y there is an odd integer x such that y^x is a perfect square. This is also false for the same reasons.

Suppose that we look instead at the statements $\exists y \forall x Q(x,y)$ and $\forall x \exists y Q(x,y)$. The first one is true; it suffices to take y to be a perfect square; if this is true then y^x will also be a perfect square. The second is true for the same reasons.∎

Note. Here is a graphical explanation of why $\exists x \forall y Q(x,y)$ is true implies that $\forall y \exists x Q(x,y)$ is true. Set up a matrix whose rows correspond to the possibilities for x and whose columns correspond to the possibilities for y; it may have infinitely many rows or columns, but that need not concern us here. Insert a T or F in each entry depending upon whether $Q(x,y)$ is true or false. Then $\forall y \exists x Q(x,y)$ means that each column has a T somewhere, and $\exists x \forall y Q(x,y)$ states that one can always find a T in some fixed row (namely, the one corresponding to x).∎

3. One way to work such a problem is to begin by listing all the integers between 1 and, say, 100. One then eliminates all the prime numbers, then all the numbers of the form $p+1$, then all numbers of the form $p+4$ and so on through $p+81$, where the constants run through all perfect squares. One then checks to see which numbers have not been eliminated as potential counterexamples. The first one on the list is 25. This is a brute force approach but it works and really does not require all that much effort.∎

4. Following the hint, write $n^3 - 1 = (n-1)(n^2+n+1)$. If the number on the left hand side is prime, then one of the two factors on the right must be equal to 1. Since we are dealing with positive numbers, it follows that $n > 0$ and hence that the second factor is greater than 1. Therefore $n - 2 = 1$, so that $n = 2$, which means that $p = n^3 - 1 = 7$.∎

5. If $3p + 1 = m^2$, then $3p = m^2 - 1 = (m-1)(m+1)$. By unique factorization into primes, one of the factors on the right must be equal to 3 and the other equal to p. If $3 = m+1$ then $m = 2$ and $p = 2$; however, $7 = 3p + 1$ is not a perfect square in this case. On the other hand, if $3 = m - 1$, then $p = 5$ and $3p + 1 = 16 = 4^2$. Therefore the only possibility is $p = 5$.∎

II.1 : Notation and first steps

Questions to answer

1. (*i*) We shall use the example of a deck of cards. Let A be the deck. Then the elements of A are single cards, and A is not a single card, so $A \notin A$.∎

(*ii*) Suppose that A is a loaf of bread, so that the elements of A are slices of bread, and let B be a shipment containing loaves of bread, including A so that $A \in B$. Then $B \notin A$ because B is not a slice of bread.■

(*iii*) Let A be a slice of the loaf of bread B, and let B be one of the loaves in shipment C. Then $A \notin C$ because it is only a slice of bread and not an entire loaf.■

2. Once again let B be a loaf of bread in shipment C, and let A be some but not all of the slices of the loaf B. Only entire loaves are elements of C, so $A \notin C$.■

3. The appropriate interpretation of a line lying on a plane is that the subset given by the line is contained in the subset given by the plane.

SOLUTIONS TO EXERCISES FOR

MATHEMATICS 144 — Part 2

Fall 2006

III. Elementary constructions on sets (*continued*)

III.2 : Ordered pairs and products

Exercises to work

1. (1) Suppose that (x, y) lies in $A \times (B \cap D)$. Then $x \in A$ and $y \in B \cap D$. Since the latter means $y \in B$ and $y \in D$, this means that

$$(x, y) \in (A \times B) \cap (A \times D) \ .$$

Now suppose that (x, y) lies in the set displayed on the previous line. Since $(x, y) \in A \times B$ we have $x \in A$ and $y \in B$, and similarly since $(x, y) \in A \times D$ we have $x \in A$ and $y \in D$. Therefore we have $x \in A$ and $y \in B \cap D$, so that $(x, y) \in A \times (B \cap D)$. Thus every element of $A \times (B \cap D)$ is also a member of $(A \times B) \cap (A \times D)$ and vice versa, and therefore the two sets are equal.■

(2) Suppose that (x, y) lies in $A \times (B \cup D)$. Then $x \in A$ and $y \in B \cup D$. If $y \in B$ then $(x, y) \in A \times B$, and if $y \in D$ then $(x, y) \in A \times D$; in either case we have

$$(x, y) \in (A \times B) \cup (A \times D) \ .$$

Now suppose that (x, y) lies in the set displayed on the previous line. If $(x, y) \in A \times B$ then $x \in A$ and $y \in B$, while if $(x, y) \in A \times D$ then $x \in A$ and $y \in D$. In either case we have $x \in A$ and $y \in B \cup D$, so that $(x, y) \in A \times (B \cap D)$. Thus every member of $A \times (B \cup D)$ is also a member of $(A \times B) \cup (A \times D)$ and vice versa, and therefore the two sets are equal.■

(3) Suppose that (x, y) lies in $A \times (Y - D)$. Then $x \in A$ and $y \in Y - D$. Since $y \in Y$ we have $(x, y) \in A \times Y$, and since $y \notin D$ we have $(x, y) \notin A \times D$. Therefore we have

$$A \times (Y - D) \ \subset \ (A \times Y) - (A \times D) \ .$$

Suppose now that $(x, y) \in (A \times Y) - (A \times D)$. These imply that $x \in A$ and $y \in Y$ but $(x, y) \notin A \times D$; since $x \in A$ the latter can only be true if $y \notin D$. Therefore we have that $x \in A$ and $y \in Y - D$, so that

$$A \times (Y - D) \ \supset \ (A \times Y) - (A \times D) \ .$$

This proves that the two sets are equal.■

(4) Suppose that (x, y) lies in $(A \times B) \cap (C \times D)$. Then we have $x \in A$ and $y \in B$, and we also have $x \in C$ and $y \in D$. The first and third of these imply $x \in A \cap C$, while the second and fourth imply $y \in B \cap D$. Therefore $(x, y) \in (A \cap C) \times (B \cap D)$ so that

$$(A \times B) \cap (C \times D) \ \subset \ (A \cap C) \times (B \cap D) \ .$$

Suppose now that (x, y) lies in the set on the right hand side of the displayed equation. Then $x \in A \cap C$ and $y \in B \cap D$. Since $x \in A$ and $y \in B$ we have $(x, y) \in A \times B$, and likewise since $x \in C$ and $y \in D$ we have $(x, y) \in C \times D$, so that

$$(A \times B) \cap (C \times D) \ \supset \ (A \cap C) \times (B \cap D) \ .$$

Therefore the two sets under consideration are equal.■

(5) Suppose that (x, y) lies in $(A \times B) \cup (C \times D)$. Then either we have $x \in A$ and $y \in B$, or else we have $x \in C$ and $y \in D$. The first and third of these imply $x \in A \cup C$, while the second and fourth imply $y \in B \cup D$. Therefore (x, y) is a member of $(A \cup C) \times (B \cup D)$ so that

$$(A \times B) \cup (C \times D) \ \subset \ (A \cup C) \times (B \cup D) \ .$$

Supplementary note: To see that the sets are not necessarily equal, consider what happens if $A \cap C = B \cap D = \emptyset$ but all of the four sets A, B, C, D are nonempty. Try drawing a picture in the plane to visualize this.■

(6) Suppose that (x, y) lies in $(X \times Y) - (A \times B)$. Then $x \in X$ and $y \in Y$ but $(x, y) \notin A \times B$. The latter means that the statement

$$x \in A \ \textbf{and} \ y \in B$$

is false, which is logically equivalent to the statement

$$\textbf{either} \ x \notin A \ \textbf{or} \ y \notin B \ .$$

If $x \notin A$, then it follows that $(x, y) \in \big((X - A) \times Y\big)$, while if $y \notin B$ then it follows that $(x, y) \in \big(X \times (Y - B)\big)$. Therefore we have

$$\big(X \times Y\big) - \big(A \times B\big) \ \subset \ \big(X \times (Y - B)\big) \cup \big((X - A) \times Y\big) \ .$$

Suppose now that (x, y) lies in the set on the right hand side of the containment relation on the displayed line. Then we have $(x, y) \in X \times Y$ and also

$$\textbf{either} \ x \notin A \ \textbf{or} \ y \notin B \ .$$

The latter is logically equivalent to

$$x \in A \ \textbf{and} \ y \in B$$

and this in turn means that $(x, y) \notin A \times B$ and hence proves the reverse inclusion of sets.■

2. $A \times B$ consists of all ordered pairs (a, b) with $a \in A$ and $b \in B$. If there are no elements in either A or B, then there is no way to make an ordered pair of this type.■

3. If the intersection is empty, then it is impossible to construct ordered pairs of the form (x, y) with $x \in A$ and $y \in B$. We claim this means that $A \cap B = \emptyset$. If not and z belongs to both, then we would have (z, z) in the intersection of the Cartesian products.■

III.3 : Larger constructions

Exercises to work

1. The set $\$(F)$ is the set of all real numbers x such that $x \leq M$ for some positive real number M. Since $|x| + 1 > 0$ and $x < 1 + |x|$, it follows that every real number x belongs to a closed interval of the form $[-M, M]$ and hence The set $\$(F)$ contains all real numbers; on the other hand, since every element of the latter is a real number, the set in question is also contained in the real numbers, so it must be equal to the real numbers. — To characterize the intersection, notice that $0 \in [-M, M]$ for all M and hence 0 lies in the intersection. On the other hand, if $x \neq 0$ then $|x| > 0$ so that

$$x \quad \notin \quad [-\tfrac{1}{2}\,|x|, \tfrac{1}{2}\,|x|\,]$$

and hence x does not lie in the intersection. Thus the intersection is the set $\{0\}$.

The same reasoning shows that one obtains the identical answers for the union and intersection if the closed intervals $[-M, M]$ are replaced by the open intervals $(-M, M)$.■

2. We need to describe the sets $L(n)$ for all integers $n > 1$. If n is even, then this set is just the open interval $(0, 1)$ because even powers of 0 and 1 are equal to 0 and 1, even powers of negative numbers are positive, while if x is positive and $n > 1$ then $x^n < x$ if and only if $x < 1$. Thus the set $L(n)$ is equal to the open interval $(0, 1)$ if n is even. Suppose now that n is odd. The preceding discussion applies equally well if x is nonnegative. Furthermore, it $x < 0$ then we know that $x^n < x$ if $x < -1$, while $(-1)^n = -1$ and $0 > x^n > x$ if $-1 < x < 0$. Thus if n is odd then $L(n)$ is just the set $(-\infty, -1) \cup (0, 1)$.

It follows that the union of the sets $L(n)$ is equal to $(-\infty, -1) \cup (0, 1)$ and the intersection of the sets $L(n)$ is equal to $(0, 1)$.

3. Suppose that X is a subset of A for all $A \in C$; then $y \in X$ implies $y \in A$ for all such A so that $y \in \cap\{A \mid A \in C\}$, and hence we also have $X \subset \cap\{A \mid A \in C\}$. Conversely, if X is a subset of $\cap_C\, A$ then $X \in \cap_C\, P(A)$, and therefore we have the equality $\cap_A\, P(A) = P(\cap_C\, A)$.

Suppose now that X is a subset of A for somew $A \in C$; then it follows that X is a subset of $\cup_C\, A$, and this yields the second relationship in the exercise.

Finally, we have that $P(\{1\}) \cup P(\{2\})$ is a proper subset of $P(\{1, 2\})$ because $\{1, 2\}$ is not a subset of either $\{1\}$ or $\{2\}$.■

4. Yes, one can use $P(A) = P(B)$ to conclude that $A = B$. Define a subset of X to be *atomic* if it has no nonempty proper subsets. Then the atomic subsets are those which contain exactly one element. If $P(A) = P(B)$, then they have the same atomic subsets. Now for one point subsets we have that $\{x\} = \{y\}$ if and only if $x = y$, and hence $x \in A$ and $P(A) = P(B)$ imply $\{x\} \in P(B)$, so that $x \in B$. It follows that $A \subset B$, and reversing the roles of A and B we also obtain $B \subset A$ so that $A = B$.■

5. The "if" direction is trivial, so we focus on the "only if" direction here. Since $(a, b, c) = (u, c)$ where $u = (a, b)$ and $(x, y, z) = (v, z)$ where $v = (x, y)$, it follows that if the ordered triples are equal then $c = z$ and $u = v$, and the latter in turn implies that $a = x$ and $b = y$.■

6. With the given definition we have $\langle x, y, x \rangle = \langle x, y, y \rangle$ even if $x \neq y$.■

III.4 : A convenient assumption

Exercises to work

1. Follow the hint. Suppose that x has Russell type k, and consider the Russell type of $\{x\}$. Since x is the only element of $\{x\}$, it follows that any $\in$-sequence

$$a_n \;\in\; a_{n-1} \;\in\; \cdots\; a_1 \;\in\; \{x\}$$

must have $a_1 = x$. If x has Russell type k then there is a sequance of this sort where $n = k+1$ but there are no sequences of this type where $n \geq k+2$. Therefore there is an $\in$-sequence for $\{x\}$ which has $k+2$ terms but no sequences with more terms, and hence the Russell type of $\{x\}$ is $k+1$.

The preceding tells us if we have a set x with Russell type zero, we also have the set $\{x\}$ with Russell type one, and likewise the singleton for the latter has Russell type two, and so on. Therefore the proof reduces to verifying that there is a set of Russell type zero, and the empty set satisfies this condition.■

2. If a set S has Russell type k, then every element of S will have Russell type at most $k-1$, and conversely if every element of S has Russell type at most $k-1$, then S has Russell type at most k.

Suppose now that A and B have finite Russell types p and q respectively and that r is the larger of p and q. Then the Russell type of $A \cup B$ is less than or equal to r.■

3. The end of any $\in$-sequence for $A \times B$ must have terms of the form

$$\cdots \{ \{a\}, \{a,b\} \} = (a,b) \in A \times B$$

and if the sequence continues then the next term down must be $\{a\}$ or $\{a,b\}$, while the term after that must be a or b. If A anad B have finite Russell type, then there is some k such that every $\in$-sequence ending with an a or a b must have at most $k+1$ terms. By the discussion above, it follows that every $\in$-sequence ending with $A \times B$ must then have at most $k+3$ terms.■

4. Every $\in$-sequence ending in $P(A)$ must end with terms of the form $b \in B \in P(A)$, where $B \subset A$. Thus if A has Russell type n, then $P(A)$ has Russell type $n+1$.■

IV. Relations and functions

IV.1 : Binary relations

Exercises to work

GENERAL REMARK. There are several exercises which ask whether a given binary relation is reflexive, symmetric, antisymmetric or transitive. We shall only work out a few representative examples in detail and give yes/no answers for the others. Details for the remaining examples appear in the handbooks written to accompany Rosen's text.

1. (a) This relation is not reflexive because $(1,1)$ and $(4,4)$ are not elements of the subset. It is not symmetric because it contains $(2,4)$ but not $(4,2)$. It is not antisymmetric because it contains $(2,3)$ and $(3,2)$, and of course $2 \neq 3$. To see it is transitive, one needs to enumerate all the pairs of ordered pairs (a,b) and (b,c) in the relation:

[1] $(2,2)$, $(2,2)$
[2] $(2,2)$, $(2,3)$
[3] $(2,2)$, $(2,4)$
[4] $(2,3)$, $(3,2)$
[5] $(2,3)$, $(3,3)$
[6] $(2,3)$, $(3,4)$

[7] $(3,2),\ (2,2)$
[8] $(3,2),\ (2,3)$
[9] $(3,2),\ (2,4)$
[10] $(3,3),\ (3,3)$
[11] $(3,3),\ (3,4)$

The transitivity of this relation amounts to saying that for each of these cases the corresponding ordered pair (a,c) lies in the relation. One checks this out on a case by case basis.∎

(b) This relation is reflexive, symmetric and transitive but not antisymmetric. We shall only give details for the first two because the others are worked in a manner similar to the previous exercise. The relation is reflexive because it contains each ordered pair (x,x). To show it is symmetric, one must list all the ordered pairs in the relation

$$(1,1),\ (2,2),\ (2,1),\ (1,2),\ (3,3),\ (4,4)$$

and check that the pairs with the entries switched

$$(1,1),\ (2,2),\ (1,2),\ (2,1),\ (3,3),\ (4,4)$$

also belong to the relation, which is straightforward.∎

(c) This relation is symmetric, but not reflexive, antisymmetric or transitive. We have already done examples for the first three types, so we shall only give details for the last conclusion. This follows because the relation contains $(2,4)$ and $(4,2)$ but neither $(2,2)$ nor $(4,4)$. If a relation is transitive and contains both $(2,4)$ and $(4,2)$, then it must also contain both $(2,2)$ and $(4,4)$.∎

(d) This relation is not reflexive, symmetric or transitive, but it is antisymmetric. The latter is **vacuously true** because the relation does not contain two ordered pairs of the form (a,b) and (b,a)!∎

(e) This relation is not reflexive, symmetric, antisymmetric or transitive. However, the statement of this problem differs from the corresponding statement in Rosen; specifically, the latter asks about the relation given by $(1,1)$, $(2,2)$, $(3,3)$, and $(4,4)$. The relation in Rosen's original problem is reflexive, symmetric, antisymmetric and transitive. We shall only discuss the antisymmetry property (problems of the other types have already been considered). The only pairs (a,b) such that both (a,b) and (b,a) are in the relation are the diagonal pairs of the form (a,a), so if one has aRb and bRa then $a=b$ follows.∎

(f) This relation is not reflexive, symmetric, antisymmetric or transitive.∎

2. (c) This relation is reflexive, symmetric and transitive, but not antisymmetric.∎

(d) This relation is not reflexive, not symmetric and not transitive, but it is antisymmetric. We shall only check the last of these. If xRy and yRx then we have $x=2y$ and $y=2x$. The only way this can happen is if $x=y=0$, and of course it does happen in this case.∎

(e) This relation is reflexive and symmetric, but it is neither antisymmetric nor transitive.∎

(f) This relation is symmetric, but it is not reflexive, not antisymmetric and not transitive.∎

3. (a) This relation is symmetric, but it is not reflexive, not antisymmetric and not transitive.∎

(b) This relation is symmetric, but it is not reflexive, not antisymmetric and not transitive.∎

(c) This relation is symmetric, but it is not reflexive, not antisymmetric and not transitive.■

(g) This relation is not reflexive, not symmetric and not transitive, but it is antisymmetric. We shall only check the last of these. If xRy and yRx then we have $x = y^2$ and $y = x^2$. Substituting the first equation into the second, we obtain $y = y^4$, so that either $y = 0$ or $y^3 = 1$. If $y = 0$ then we must also have $x = 0$. If $y^3 = 1$ and we are dealing with real numbers, then we must have $y = 1$, which in turn implies $x = 1$. [*Note:* If we allowed complex numbers then the relation would not be antisymmetric, for if x is a non-real cube root of 1 and $y = x^2$ then $y \neq x$ but $x = y^2$.]■

(h) This relation is not reflexive, not symmetric, not transitive, and not antisymmetric. We shall only check the last two of these. A counterexample to transitivity is given by $(x, y) = (\frac{1}{3}, \frac{1}{2})$ and $(y, z) = (\frac{1}{2}, \frac{3}{5})$. For these choices we have $x > y^2$ and $y > z^2$ but $x < z^2$. A counterexample to antisymmetry is given by the same (x, y).■

4. Yes. For the reflexive property, if for each x we have $(x, x) \in R_1$ and $(x, x) \in R_2$, then we also have $(x, x) \in R_1 \cap R_2 \subset R_1 \cup R_2$. For the symmetry property, suppose first that $(x, y) \in R_1 \cap R_2$. Then $(x, y) \in R_1$ and $(x, y) \in R_2$, and since R_1 and R_2 are symmetric it follows that $(y, x) \in R_1$ and $(y, x) \in R_2$, so that $(y, x) \in R_1 \cap R_2$. Suppose now that that $(x, y) \in R_1 \cup R_2$. Then $(x, y) \in R_1$ or $(x, y) \in R_2$, and since R_1 and R_2 are symmetric it follows that $(y, x) \in R_1$ or $(y, x) \in R_2$ respectively, so that $(y, x) \in R_1 \cap R_2$.■

5. (a) Equivalence relation.■

(b) Not reflexive and not transitive.■

(c) Equivalence relation.■

(d) Not transitive.■

(e) Not symmetric and not transitive.■

6. (a) Equivalence relation.■

(b) Equivalence relation.■

(c) Not transitive.■

(d) Not transitive.■

(e) Not transitive.■

7. The relation S is reflexive, for R is reflexive and xRx and xRx imply xSx. Suppose now that xSy. Then xRy and yRx, and by definition this also implies ySx. Finally, suppose that xSy and ySz. Then we have xRy and yRx and, we also have yRz and zRy. By the transitivity of R these imply that xRz and zRx, which means that xSz. Therefore S is an equivalence relation.■

8. We first prove the ($\Longrightarrow$) implication. Suppose that R is an equivalence relation. Then it is automatically reflexive. Suppose now that xRy and yRz. Then we also have xRz because R is transitive. But since R is symmetric the latter implies zRx and hence T is circular. Now we prove the ($\Longleftarrow$) implication. By assumption R is reflexive. To show that it is symmetric, suppose that xRy. If we combine this with xRx (since R is reflexive) and the circular property we conclude that yRx. Finally, if xRy and yRz, then zRx since R is circular. However, we have shown that R is symmetric, and therefore we also have xRz so that R is transitive. Hence R is an equivalence relation.■

9. We have $(x, y)P(x, y)$ because $xy = xy$, and if $(x, y)P(z, w)$ then $xw = yz$, which is equivalent to $zy = wx$, which means that $(z, w)P(x, y)$. Finally, if $(x, y)P(z, w)$ and $(z, w)P(u, v)$, then $xw = yz$ and $zv = uw$. Multiplying these equations together yields $xwzv = yzuw$, and since $w \neq 0$ it follows that $xzv = yzu$. If $z \neq 0$ then we may divide both sides of the equation by z and obtain $xv = yu$, which implies $(x, y)P(u, v)$.

Suppose now that $z = 0$. Then $0 = yz = xw$ and since $w \neq 0$ it follows that $x = 0$. Likewise, $0 = zv = uw$ implies $u = 0$. But then we have $xv = 0v = 0 = 0w = uw$, so that $(x, y)P(u, v)$ in this case too. Therefore we have shown that the relation is an equivalence relation.

To prove the final assertion, we first show there is at least one r such that $(x, y) = (r, 1)$. Specifically, if $r = x/y$, then $yr = x = x \cdot 1$. Next, we must show there is only one such r, so suppose we have $(x, y)P(s, 1)$. Then by the definition of the equivalence relation we have $ys = x$, so that $s = x/y$, and hence s must be equal to the value of r give previously.■

10. Taking logarithms, we find that $(x, y)Q(z, w)$ if and only if $w \ln x = y \ln z$. One can now proceed as in the previous exercise to show that Q is reflexive and symmetric, and also that Q is transitive with separate consideration for the cases $\ln z \neq 0$ and $\ln z = 0$. Therefore the relation Q is also an equivalence relation.■

11. (*i*) There is a figure to illustrate the argument in the file `knightmoves.JPG`, with the knight starting at its usual position at time 1 and different colors indicating new possibilities for its positions at times 2 through 6. The picture indicates that the knight can reach every square in 6 moves or less. Writing everything down in detail is left to the reader.■

(*ii*) The figure `knight2.JPG` shows how a knight starting at $(0, 0)$ can reach the diagonally adjacent square $(-1, -1)$ after two moves and the horizontally adjacent square $(1, 0)$ after three moves, with the first move going to $(1, 2)$. Symmetry considerations yield all the eight cases as follows:

$$(0,0) \to (1,2) \to (-1,1) \to (1,0)$$

$$(0,0) \to (1,-2) \to (-1,-1) \to (1,0)$$

$$(0,0) \to (-1,-2) \to (1,1) \to (-1,0)$$

$$(0,0) \to (-1,2) \to (1,-1) \to (-1,0)$$

$$(0,0) \to (2,1) \to (1,-1) \to (0,1)$$

$$(0,0) \to (-2,1) \to (-1,-1) \to (0,1)$$

$$(0,0) \to (-2,-1) \to (1,1) \to (0,-1)$$

$$(0,0) \to (2,-1) \to (-1,1) \to (0,-1)$$

Note that there is duplication in this list, for each adjacent square appears in exactly two sequences.■

12. The union of the relations is that a is a multiple of b or b is a multiple of a, and the intersection is that a is a multiple of b and b is a multiple of a. The first of these cannot be simplified, but the second can as follows: if $a = xb$ and $b = ya$, then $a = xb = xya$ implies that $xy = 1$, so that $x = y = \pm 1$, and hence the intersection is the relation that $a = \pm b$.■

13. As noted in the hint, the statement $x\ S \circ T\ y$ means that $y - 2 = b(ax - 1)$ where a and b are ± 1. There are two sign choices for each of a and b, and since they may vary independently there are a total of four possible values of y related to a given value of x under the relation $S \circ T$:

$$y - 2 = 2 + (x - 1) = x + 1$$

$$y - 2 = 2 + (-x - 1) = 1 - x$$

$$y - 2 = 2 - (x - 1) = 3 - x$$

$$y - 2 = 2 - (-x - 1) = 3 + x$$

If $x = 1$ we obtain the possible values of $2, 0, 4$ for y; note that the first and third formulas give the same value for y. On the other hand, If $x = 2$ we obtain the possible values of $3, -1, 1, 5$ for y.■

14. Suppose that $x \; S\circ(T_1 \cup T_2) \; y$. Then we have $x \; S \; z$ and $z \; (T_1 \cup T_2) \; y$ for some z. If $z \; T_1 \; y$ then we have $x \; S\circ T_1 \; y$ and on the other hand if $z \; T_2 \; y$ then we have $x \; S\circ T_2 \; y$; in both cases we have

$$x \; [S\circ T_1 \; \cup \; S\circ T_2] \; y$$

and therefore we have $S\circ(T_1 \cup T_2) \subset S\circ T_1 \cup S\circ T_2$. — Conversely, suppose that $x \; [S\circ T_1 \; \cup \; S\circ T_2] \; y$. Then either there is some z such that $x \; S \; z$ and $z \; T_1 \; y$ or else there is some z such that $x \; S \; z$ and $z \; T_2 \; y$. In both cases we have $z \; (T_1 \cup T_2) \; y$ and hence $x \; S\circ(T_1 \cup T_2) \; y$.

Suppose now that $x \; S\circ(T_1 \cap T_2) \; y$. Then we have $x \; S \; z$ and $z \; (T_1 \cap T_2) \; y$ for some z. In particular, we have $z \; T_1 \; y$ and $z \; T_2 \; y$, which means that

$$x \; [S\circ T_1 \; \cap \; S\circ T_2] \; y$$

and consequently $S\circ(T_1 \cap T_2) \subset S\circ T_1 \cap S\circ T_2)$.

Here is an example where the containment in the preceding paragraph is proper. Take a set consisting of the four objects $\{x, a, b, y\}$, and define binary relations S, T_1, T_2 such that the following are the only true statements about them:

$$x \; S \; a, \qquad x \; S \; b, \qquad a \; T_1 \; y, \qquad b \; T_2 \; y$$

Then $T_1 \cap T_2 = \emptyset$, and hence we must also have $S\circ(T_1 \cap T_2) = \emptyset$; in other words, there are no choices of u and v for which $u \; S\circ(T_1 \cap T_2) \; v$ is true. On the other hand, by construction we know that $x \; S\circ T_1 \; y$ and $x \; S\circ T_2 \; y$ are both true, so that

$$x \; [S\circ T_1 \; \cap \; S\circ T_2] \; y$$

is true, Therefore the binary relations $S\circ T_1 \; \cap \; S\circ T_2$ and $S\circ(T_1 \cap T_2)$ are not equal.■

SOLUTIONS TO EXERCISES FOR

MATHEMATICS 144 — Part 3

Fall 2006

IV. Relations and functions

IV.2 : Partial and linear orderings

Exercises to work

1. We shall prove the statement about intersections of partial orderings first. By definition we have $x\ (P_1 \cap P_2)\ y$ if and only if $x\ P_1\ y$ and $x\ P_2\ y$. REFLEXIVE PROPERTY. Since $x\ P_1\ x$ and $x\ P_2\ x$, we have $x\ (P_1 \cap P_2)\ x$. SYMMETRIC PROPERTY. In this case we are given $x\ P_1\ y$, $x\ P_2\ y$, $y\ P_1\ x$ and $y\ P_2\ x$. Since P_1 and P_2 are both partial orderings it follows that $x = y$. TRANSITIVE PROPERTY. We are now given $x\ P_1\ y$, $x\ P_2\ y$, $y\ P_1\ z$ and $y\ P_2\ z$. Since P_1 and P_2 are both partial orderings it follows that $x\ P_1\ z$ and $x\ P_2\ z$, so that $x\ (P_1 \cap P_2)\ z$.

Finally, we need to show that $P_1 \cup P_2$ is not a partial ordering. Take a set with two elements a and b, and let P_1 and P_2 be the unique partial orderings for which $a\ P_1\ b$ and $b\ P_2\ a$. Then $a \neq b$ but $a\ (P_1 \cup P_2)\ b$ and $b\ (P_1 \cup P_2)\ a$, so the union relation is not antisymmetric and hence it cannot be a partial ordering.■

Further question. If the two orderings in the previous problem are linear, then the intersection is not necessarily a linear ordering (look at the second example; in this case a and b are not comparable).■

2. The elements greater than $(2, 3)$ in the lexicographic ordering are $(2, 4)$, $(3, n)$ and $(4, n)$ where $n = 1, 2, 3, 4$. Likewise, the elements less than $(3, 1)$ are $(1, n)$ and $(2, n)$ for the same range of values for n.■

3. We shall work the parts in order.

(1) The reflexive property follows from the construction. To prove the symmetric property, suppose that $[a, b]\ P\ [c, d]$ and $[c, d]\ P\ [a, b]$. If $[a, b] \neq [c, d]$, then by definition we have $b < c$ and $d < a$. Since $a < b$ and $c < d$, this yields the impossible chain of strict inequalities $a < b < c < d < a < b$. Thus the only logical possibility is for the two intervals to be equal. Finally, suppose we have $[a, b]\ P\ [c, d]$ and $[c, d]\ P\ [e, f]$. The conclusion is trivial if either $[a, b] = [c, d]$ or $[c, d] = [e, f]$, so let us assume that neither holds. We then have $b < c < d < e < f$, and this implies $[a, b]\ P\ [e, f]$.

(2) The statement of the exercise is equivalent to the statement that if $[a, b]$ and $[c, d]$ are comparable but unequal, then they must be disjoint. Suppose the two intervals are unequal. Then if $[a, b]\ P\ [c, d]$, we have $b < c$ which means that the two intervals are disjoint, while if $[c, d]\ P\ [a, b]$, we have $d < a$ which also means that the two intervals are disjoint.

(3) Consider the intervals $[0, 1]$ and $[0, 2]$. These intervals are not equal and not disjoint, so by the preceding part of the exercise they cannot be comparable with respect to the partial ordering P.■

4. If we have a linearly ordered chain $S_1 < \cdots < S_m$ then the number of elements in S_k is at least one more than the number of elements in S_{k-1}. Since S_1 contains at least zero elements, this means that the number of elements in S_m is at least $m-1$. Since S has n elements, this means that $m-1 \leq n$ or $m < n+2$. To get a linearly ordered set with $n+1$ elements, start with $S_1 = \emptyset$, and for $1 < k \leq n+1$ let $S_k = \{1, \cdots, k-1\}$.■

5. The relation is reflexive because $p(x) \leq p(x)$ for all x. It is antisymmetric because $p(x) \leq q(x)$ for all x and $q(x) \leq p(x)$ for all x imply $p(x) = q(x)$ for all x, and hence $p = q$. It is transitive because $p(x) \leq q(x)$ for all x and $q(x) \leq r(x)$ for all x imply $p(x) \leq r(x)$ for all x, so that $p \leq r$.

To show this partial ordering is not a linear ordering we need to find two polynomials p and q such that p is not less than or equal to q and vice versa. It will be enough to find p and q together with real numbers a and b such that $p(a) > q(a)$ but $p(b) < q(b)$, for the first implies $p \leq q$ is false and the second implies that $p \geq q$ is false.

Specifically, take p to be the constant polynomial with value 1 and let $q(x) = x$. Then we have $q(1) > p(1)$ but $q(-1) < p(-1)$.■

6. (a) ℓ and m are the maximal elements.■

(b) a, b and c are the minimal elements.■

(c) No. There is no element that is greater than or equal to both ℓ and m.■

(d) There is no least element, because there is no element that is less than or equal to all of a, b, c.■

(e) ℓ and m are the upper bounds.■

(f) There is none; such an element would have to be less than or equal to both of the upper bounds described above, and no such upper bound exists.■

(g) There are no elements that are less than or equal to all three of f, g, h.■

(h) Since there is no lower bound, there cannot be a greatest lower bound.■

15. There are two things two prove, the first being the general fact about betweenness and the second being the assertion that exactly one element lies between the other two. To prove the first, note that y is between x and z means that $x < y < z$ or $z < y < x$, and this is equivalent to saying that $z < y < x$ or $x < y < y$, which is the condition for y to be between z and x.

There are six different ways of permuting the variables x, y, z in the statement, "x is between y and z." By the previous paragraph they can be grouped into three pairs such that the two statements in each pair are logically equivalent:

(1) "y is between x and z" and "y is between z and x."
(2) "z is between x and y" and "z is between y and x."
(3) "x is between y and z" and "x is between z and y."

Given an arbitrary subset of three distinct elements, we need to prove that one of these pairs of statements will be true and the others will be false. The discussion splits into several cases.

Case 1. Suppose that $x < y$. Then there are subcases depending upon z is related to x and y. SUBCASE 1A. $y < z$. — In this case y is between x and z. SUBCASE 1B. $z < y$ and $x < z$. — In this case z is between x and y. SUBCASE 1C. $z < y$ and $z < x$. — In this case x is between z and y.

Case 2. Suppose that $y < x$. There are again subcases depending upon z is related to x and y. SUBCASE 2A. $z < y$. — In this case y is between x and z. SUBCASE 2B. $y < z$ and $z < x$. — In this case z is between x and y. SUBCASE 2C. $y < z$ and $x < z$. — In this case x is between z and y.

In every case we have shown that one element is between the other two. We shall conclude by showing that if y is between x and z, then z is not between y and x and x is not between y and z. If we can show this, we are essentially done, for the other two cases will follow by interchanging the roles of x, y and z in the argument.

Suppose first that y is between x and z and z is also between y and x. Then we have $x < y < z$ or else we have $z < y < x$. Similarly, we also have $y < z < x$ or $x < z < y$. There are four possible pairs of inequalities under the given assumptions. We shall show they all lead to contradictions.

(1) $x < y < z$ and $y < z < x$ lead to the conclusion $x < x$, which we know is false.

(2) $x < y < z$ and $x < z < y$ are inconsistent because only one of the relations $y < z$ and $z < y$ can be true.

(3) $z < y < x$ and $y < z < x$ again lead to the conclusion $x < x$, which we know is false.

(4) $z < y < x$ and $x < z < y$ are again inconsistent because only one of the relations $y < z$ and $z < y$ can be true.

Therefore we have shown that if y is between x and z, then neither x nor z can be between the remaining two points, and as noted before this suffices to complete the proof.■

IV.3 : Functions

Exercises to work

1. There are two reasons why it is not the graph of a function. The first is that Grover Cleveland served nonconsecutive terms and was succeeded by Benjamin Harrison after the first term and by William McKinley after the second. The other reason is that the successor to the current President is not presently known,■

2. The main point is to find the graph. If A is nonempty, then we simply take its graph to be $A \times \{x\}$; this represents the constant function whose value is x at every element of A. If A is empty, then we take the graph to be the empty set. These prove existence of a function from A to $\{x\}$.

To prove the function is unique, if A is empty, then its graph is a subset of $A \times \emptyset = \emptyset$ and thus is equal to the graph of the previously defined example. If A is not empty, then $A \times \{x\}$ is the only subset of itself that contains a point with first coordinate a for all choices of $a \in A$, and therefore the graph of an arbitrary function from A to $\{x\}$ must be equal to that of the example in the previous paragraph.■

3. It is vacuously true that $\emptyset = \emptyset \times X$ is the graph of a function from $\emptyset$ to X, and conversely since every subset of $\emptyset \times X$ is empty it follows that there is only one possibility for the graph.

On the other hand, if X is not empty then $X \times \emptyset = \emptyset$, and hence every subset of $X \times \emptyset$ is also empty; therefore if $x \in X$ is arbitrary then there is no ordered pair of the form (x, y) in a subset of $X \times \emptyset$, so no subset of the latter can be the graph of a function.■

4. The domain is the set of all positive integers, and the range is the set

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \ .\blacksquare$$

5. The domain is the set of all positive integers, and the range is the set of integers

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8\} \ .$$

Note that at least one of the digits from 1 through 9 must appear, and any higher number can also appear.■

6. (*i*) We need to solve the equation $f(x) = 3$, where $f(x) = 3x - 7$. But if $3x - 7 = 3$, then $x = \frac{10}{3}$, and hence the inverse image is $\{\frac{10}{3}\}$.■

(*ii*) We need to find the singleton set whose element is $f(5) = 3 \cdot 5 - 7 = 8$. Therefore the set we want is $\{8\}$.■

(*iii*) We need to find all x such that $-7 \leq 3x - 7 \leq 2$. Adding 7 to everything in sight we see that the inequalities are equivalent to $0 \leq 3x \leq 9$, which in turn is equivalent to $x \in [0, 3]$. Hence in this case the inverse image is equal to $[0, 3]$.■

(*iv*) We need to find the set of all y such that $y = 3x - 7$ for $2 \leq x \leq 6$. This turns out to be the interval $[-1, 111$, which is thus the image of the given interval.■

(*v*) The image of the empty set is always equal to the empty set.■

(*vi*) We need to find all x such that $3 \leq 3x - 7 \leq 5$ or $3x - 7 \leq 8 \leq 10$, and these are equivalent to $10 \leq 3x \leq 12$ or $15 \leq 3x \leq 17$. Thus in this case the inverse image is equal to $[\frac{10}{3}, 4] \cup [5, \frac{17}{3}]$.■

7. (*i*) This is just the singleton set containing $f(-1) = 0$.■

(*ii*) The inequalities $0 \leq (x + 1)^2 \leq 1$ hold if and only if $-1 \leq x + 1 \leq 1$, which in turn is equivalent to $x \in [-2, 0]$, so $[-2, 0]$ is the inverse image of the set in the problem.■

(*iii*) Since $(x + 1)^2 \geq 0$ for all x, the inverse images of $[-1, 1]$ and $[0, 1]$ are equal, so the answer is the same as for the previous problem.■

(*iv*) The inequalities $-3 \leq (x + 1)^2 \leq 5$ are equivalent to $0 \leq (x + 1)^2 \leq 5$, which in turn is equivalent to $-5 \leq x + 1 \leq 5$, so that the inverse image in this case is equal to $[-6, 4]$.■

(*v*) The inverse image of $[-3, -1]$ is empty because f takes nonnegative values. As in (*v*) from the previous exercise, this implies that the set to be determined is the empty set.■

(*vi*) By (*iii*) we know that the inverse image of $[-1, 1]$ is equal to $[-2, 0]$, and the image of the latter under f is equal to $[0, 1]$, because $x \in [-2, 0]$ implies $x + 1 \in [-1, 1]$, so that $0 \leq (x + 1)^2 \leq 1$.■

SOLUTIONS TO EXERCISES FOR

MATHEMATICS 144 — Part 3

Fall 2006

IV. Relations and functions

IV.4 : Composite and inverse functions

Exercises to work

1. Suppose that X is linearly ordered and $a, b \in X$ are distinct. Then either $a < b$ or $b < a$, and since f is strictly increasing this means that $f(a) < f(b)$ or $f(b) < f(a)$. In both cases we have $f(a) \neq f(b)$, and therefore f is 1–1.

Here is a counterexample when X is not linearly ordered. Let X be the set of all subsets of $\{0, 1\}$ ordered by inclusion, let Y be the nonnegative integers, and let $f : X \to Y$ denote the number of elements in the subset $A \subset \{0, 1\}$. Then f is strictly increasing, but $f[\{0\}] = f[\{1\}]$.■

2. First of all, the problem should be corrected to read, "Given a set X, let $P_0(X)$ denote the set of **nonempty** subsets of X, and define $h : P_0(A) \times P_0(B) \to P_0(A \times B)$ by $h(C, D) = C \times D$."

[Otherwise the map is not 1–1 because, say, $h(\emptyset, D) = \emptyset$.]

If $h_0(C, D) = h_0(C', D')$ then $C \times D = C' \times D'$. Suppose that $x \in C$ and $y \in D$. Then $(x, y) \in C \times D = C' \times D'$ implies that $x \in C'$ and $y \in D'$, so that $C \subset C'$ and $d \subset D'$. Conversely, if $x \in C'$ and $y \in D'$, then $(x, y) \in C' \times D' = C \times D$ implies that $x \in C'$ and $y \in D'$, so that $C \subset C'$ and $d \subset D'$. Therefore $C = C'$ and $D = D'$. To see that h_0 is not onto, let $A = B = \{0, 1\}$ and note that $E = A \times B - \{(1, 1)\}$ is not in the image of h_0, which consists of the sets $\{x, y)\}$, $\{x\} \times B$, $A \times \{y\}$, and $A \times B$. Note that there are 9 sets in the image and 15 sets in the codomain.■

3. (a) This map is injective.■

(b) This map is not injective.■

(c) This map is not injective.■

4. (a) This map is bijective.■

(b) This map is not bijective; it is neither injective nor surjective.■

(c) This map is not bijective from the reals to themselves, but it does define a bijection from $\mathbf{R} - \{-2\}$ to $\mathbf{R} - \{1\}$.■

(d) This map is bijective.■

5. Suppose that f is a formal monomorphism; we need to show that f is also injective. In other words, if $x \neq y$ we need to prove that $f(x) \neq f(y)$. Let $X, Y : \{1\} \to A$ be the functions such that $X(1) = x$ and $Y(1) = y$. Then $f \circ X(1) = f(x)$ and $f \circ Y(1) = f(y)$. It suffices to show that $f \circ X \neq f \circ Y$. Assume the contrary, so that $f \circ X = f \circ Y$. Since f is a formal monomorphism this would imply that $X = Y$, which would mean that $x = y$, a contradiction. Therefore a formal monomorphism is injective.

Conversely, suppose that f is injective, and let $g, h : C \to A$ satisfy $f \circ g = f \circ h$. Then for all $z \in C$ we have $f(g(z)) = f(h(z))$, and since f is injective this means that $g(z) = h(z)$ for all z. Therefore $g = h$; since the latter were arbitrary pairs of functions satisfying $f \circ g = f \circ h$, it follows that f is a formal monomorphism.■

6. Suppose that f is surjective, and suppose that $g, h : B \to D$ satisfy $g \circ f = h \circ f$; we want to show that $g = h$. Let $b \in B$, and choose $a \in A$ such that $f(a) = b$. Then $g(b) = g(f(a)) = h(f(a)) = h(b)$, and since b was an arbitrary element of B it follows that $g = h$. Therefore f is a formal epimorphism.

To prove the other implication, suppose that f is not surjective. Define $g, h : B \to \{0, 1\}$ as follows: Set $g(b) = 1$ for all b, and set $h(b) = 1$ if $b = f(a)$ for some a and set $h(b) = 0$ otherwise. Then $g \neq h$ because f is not surjective, but for all $a \in A$ we have $g(f(a)) = h(f(a)) = 1$ so that $g \circ f = h \circ f$, and hence f is not a formal epimorphism.■

7. (*b*) We always have $f[A \cap B] \subset f[A] \cap f[B]$, so we really need to prove that $f[A \cap B] \supset f[A] \cap f[B]$ for all A and B if and only if f is 1–1. ($\Longrightarrow$) Let $A = \{a\}$ and $B = \{b\}$ where $A \neq B$. Then $A \cap B = \emptyset$, so the condition on images implies

$$\emptyset \quad = \quad f[A \cap B] \quad \supset f[A] \cap f[B] \quad = \quad \{f(a)\} \cap \{f(b)\}$$

which can only happen if $f(a) \neq f(b)$. Thus f is 1–1 if $f[A \cap B] \supset f[A] \cap f[B]$ for all A and B. ($\Longleftarrow$) Suppose now that $f[A \cap B] \not\supset f[A] \cap f[B]$ for some A and B. Note first that both A and B must be nonempty because $f[A \cap B] = f[A] \cap f[B]$ if either A or B is empty. Then by noncontainment there is some $y \in f[A] \cap f[B]$ such that $y \not\in f[A \cap B]$. This means there are $a \in A$ and $b \in B$ such that $y = f(a) = f(b)$, but $a \not\in B$ and $b \not\in A$. The preceding conditions mean that $a \neq b$, and since $f(a) = f(b)$ it follows that f is not 1–1.■

(*b*) ($\Longrightarrow$) Let $A = \{a\}$ and let $b \neq a$. Then $f(b) \in f[X - A] \subset Y - f[A] = Y - \{f(a)\}$ implies that $f(b) \neq f(a)$, and therefore f is 1–1. ($\Longleftarrow$) Suppose now that there is a subset $A \subset X$ such that $f[X - A] \not\subset Y - f[A]$. Note first that $A \neq \emptyset$ because we do have set-theoretic containment (in fact, equality!) if $A = \emptyset$. It follows that there is some y such that $y \in f[X - A]$ but $y \not\in Y - f[A]$. The second condition is equivalent to $y \in f[A]$. We have thus shown that there are $u, v \in X$ such that $u \in A$ and $v \not\in A$ but $y = f(u) = f(v)$. Therefore f is not 1–1.■

(*d*) ($\Longrightarrow$) Let $y \in Y$. If $y \in f[A]$ then $y = f(a)$ for some $a \in A$, while if $y \in Y - f[A] = f[X - A]$ then $y = f(b)$ for some $b \in X - A$. In both cases y lies in the image of f. ($\Longleftarrow$) Suppose that f is onto. If $y \in Y - f[A]$, then $y = f(x)$ for some x, but since $y \not\in f(A)$ it follows that $x \in X - A$ and hence $y \in\subset f[X - A]$. Therefore $Y - f[A] \subset f[X - A]$.■

8. Follow the hint. Let $z \in A$ be arbitrary, and define $g : B \to A$ in cases: If $b = f(a)$ for some $a \in A$, set $g(b) = a$; since f is 1–1 there is only one possible choice for a, so this defines the function on all points of the given type. If $b \neq f(a)$ for any choice of a, then set $g(b) = z$. Then $g(f(a)) = a$ by construction, so that $g \circ f = \mathrm{id}_A$.■

9. Suppose we are given $h, k : C \to A$ such that $f \circ h = f \circ k$. Compose both sides with g:

$$g \circ f \circ h \quad = \quad \mathrm{id}_A \circ h \quad = \quad h$$

$$g \circ f \circ k \quad = \quad \mathrm{id}_A \circ k \quad = \quad k$$

Since $f \circ h = f \circ k$ the expressions on the left hand sides of both lines are equal, and hence the same is true for the expressions on the right hand sides. Hence $h = k$ and f is a monomorphism.

To show that g is an epimorphism, suppose we are given $u, v : B \to D$ such that $u \circ g = v \circ g$, and compose both sides with f:

$$u \circ g \circ f \quad = \quad u \circ \mathrm{id}_A \quad = \quad u$$

$$v \circ g \circ f \quad = \quad v \circ \mathrm{id}_A \quad = \quad v$$

As before the left hand sides of both lines are equal, so the right hand sides are too, and hence $u = v$ so that g is an epimorphism.■

10. This is really the same as the previous exercise if we interchange the roles of A and B and of f and g. The whole point is that we have two maps whose composite QP is the identity, and under these conditions P is a monomorphism and Q is an epimorphism.■

11. The simplest example is the linear mapping $L(t)$ sending $t \in [0,1]$ to $a + t(b-a)$. The mapping is 1–1 because $L(s) = L(t)$ implies $a + s(b-a) = a + t(b-a)$ and since $a \neq b$ one can solve this equation to conclude that $s = t$. To see that the mapping is onto, one need only check that if $a \leq c \leq b$ and

$$t \quad = \quad \frac{c-a}{b-a}$$

then $0 \leq t \leq 1$ and $L(t) = c$. The 1–1 correspondence L is definitely not unique. Let K be any 1–1 correspondence of $[0,1]$ with itself that is not the identity; for example, one could take $K(t) = t^2$, whose inverse is the square root function. Then $L \circ K$ is a different 1–1 correspondence between $[0,1]$ and $[a,b]$.■

12. Let $\mathbf{N}$ be the set of nonnegative integers, let $f : \mathbf{N} \to \mathbf{N}$ send x to $x+1$, and let $g : \mathbf{N} \to \mathbf{N}$ send x to $|x-1|$. Then f is not surjective because $0 \neq f(x)$ for any x, and g is not injective because $g(2) = g(0)$, but $g \circ f$ is the identity on $\mathbf{N}$. This yields the example for the first part of the problem.

As noted there are four subcases to the second part of the problem:

Is f injective if $g \circ f$ is bijective? YES. Suppose that x and y lie in the domain of f and $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$, and since $g \circ f$ is bijective it follows that $x = y$.

Is f surjective if $g \circ f$ is bijective? NOT NECESSARILY. Consider the example constructed for the first part of the problem.

Is g surjective if $g \circ f$ is bijective? YES. If z is in the codomain of g, then the bijectivity of $g \circ f$ implies that $z = g(f(w))$ for some w, and thus we know that g maps $f(w)$ to z.

Is g injective if $g \circ f$ is bijective? NOT NECESSARILY. Consider the example constructed for the first part of the problem.■

13. In both cases the idea is to write $y = f(x)$ and solve for x in terms of y.

If $f(x) = 3x - 1$, then $y = 3x - 1$ implies that $x = \frac{1}{3}(x+1)$, so the right hand side gives the inverse function.

If $f(x) = x/(1+|x|)$, then solving for x in terms of y splits into two cases depending upon whether $x \geq 0$ or $x \leq 0$. Note that these are equivalent to $y \geq 0$ and $y \leq 0$ by the definition of f. If $x \geq 0$ then we have

$$y \quad = \quad \frac{x}{1+x} \quad \Longrightarrow \quad x \quad = \quad \frac{y}{1-y}$$

while if $x \leq 0$ we have

$$y \quad = \quad \frac{x}{1+x} \quad \Longrightarrow \quad x \quad = \quad \frac{y}{1+y}$$

so in either case we have

$$x \quad = \quad \frac{y}{1-|y|}$$

as the formula for the inverse function.■

14. Let $n = \mathbf{int}(x)$. There are two cases depending upon whether $n \leq x < n + \frac{1}{2}$ or $n + \frac{1}{2} \leq x < n + 1$.

In the first case we have $\mathbf{int}(x) = \mathbf{int}\left(x + \frac{1}{2}\right) = n$ and since $2n \leq 2x < 2n + 1$ we also have $\mathbf{int}(2x) = 2n$. Therefore $\mathbf{int}(2x)$ and $\mathbf{int}(x) + \mathbf{int}\left(x + \frac{1}{2}\right)$ are both equal to $2n$ in this case.

In the second case we have $\mathbf{int}(x) = n$ but $\mathbf{int}(x + \frac{1}{2}) = n + 1$ and $2n + 1 \leq 2x < 2n + 2$. Therefore $\mathbf{int}(2x)$ and $\mathbf{int}(x) + \mathbf{int}\left(x + \frac{1}{2}\right)$ are both equal to $2n + 1$ in this case.■

15. Both statements are false, and this can be seen by taking $x = y = \frac{3}{5}$. For these choices we have

$$1 \quad = \quad \mathbf{int}\left(\tfrac{3}{5} + \tfrac{3}{5}\right) \quad \neq \quad \mathbf{int}\left(\tfrac{3}{5}\right) \; + \; \mathbf{int}\left(\tfrac{3}{5}\right) \quad = \quad 0 \; + \; 0 \quad = \quad 0$$

and

$$2 \quad = \quad \mathbf{int}\left(\tfrac{6}{5}\right) \; + \; \mathbf{int}\left(\tfrac{6}{5}\right) \quad \neq \mathbf{int}\left(\tfrac{3}{5}\right) \; + \; \mathbf{int}\left(\tfrac{3}{5}\right) \; + \; \mathbf{int}\left(\tfrac{6}{5}\right) \quad = \quad 0 \; + \; 0 \; + \; 1 \quad = \quad 1\;.$$

Therefore the have values of x and y for which $\mathbf{int}(x) + \mathbf{int}(y) \neq \mathbf{int}(x + y)$ and $\mathbf{int}(x) + \mathbf{int}(y) + \mathbf{int}(x + y) \neq \mathbf{int}(2x) + \mathbf{int}(2y)$.■

16. Let $n = \mathbf{int}(x)$. There are three cases depending upon whether $n \leq x < n + \frac{1}{3}$ or $n + \frac{1}{3} \leq x < n + \frac{2}{3}$ or $n + \frac{2}{3} \leq x < n + 1$.

In the first case we have $\mathbf{int}(x) = \mathbf{int}\left(x + \frac{1}{3}\right)) = \mathbf{int}\left(x + \frac{2}{3}\right) = n$ and since $3n \leq 3x < 3n + 1$ we also have $\mathbf{int}(3x) = 3n$. Therefore $\mathbf{int}(3x)$ and $\mathbf{int}(x) + \mathbf{int}\left(x + \frac{1}{3}\right) + \mathbf{int}\left(x + \frac{2}{3}\right)$ are both equal to $3n$ in this case.

In the second case we have $\mathbf{int}(x) = n$ and $\mathbf{int}\left(x + \frac{1}{3}\right) = n$ but $\mathbf{int}\left(x + \frac{2}{3}\right) = n + 1$ and $3n + 1 \leq 3x < 3n + 2$. Therefore $\mathbf{int}(3x)$ and $\mathbf{int}(x) + \mathbf{int}\left(x + \frac{1}{3}\right) + \mathbf{int}\left(x + \frac{2}{3}\right)$ are both equal to $3n + 1$ in this case.

In the third case we have $\mathbf{int}(x) = n$ but $\mathbf{int}\left(x + \frac{1}{3}\right) = \mathbf{int}\left(x + \frac{2}{3}\right) = n + 1$ and $3n + 2 \leq 3x < 3n + 3$. Therefore $\mathbf{int}(3x)$ and $\mathbf{int}(x) + \mathbf{int}\left(x + \frac{1}{3}\right) + \mathbf{int}\left(x + \frac{2}{3}\right)$ are both equal to $3n + 2$ in this case.■

17. The important point is that if a and b are positive real numbers, then

$$a \quad < \quad b \quad \Longleftrightarrow \quad \frac{1}{a} \quad > \quad \frac{1}{b}\;.$$

Suppose now that f is strictly increasing. Then $x < y$ implies $f(x) < f(y)$, and by the preceding line and $g = 1/f$ we conclude that $g(x) > g(y)$, so that g is strictly decreasing. Conversely, if g is strictly decreasing and $x < y$, then we have $g(x) > g(y)$. Now $g = 1/f$ is true if and only if $f = 1/g$, and therefore we can use the displayed statement to conclude that $f(x) < f(y)$ and hence that f is strictly increasing.■

18. Follow the hints. Given $y \in C_0$ we would like to define H by choosing $x \in A$ so that $q_0(x) = y$ [such an x exists because q_0 is onto] and setting $H(y) = q_1(x)$.

In order to make such a definition it is necessary to show that the construction does not depend upon the choice of x; in other words, if $q_0(z) = q_0(x) = y$, then $q_1(x) = q_1(z)$. All we have at our disposal are the injectivity and surjectivity assumptions along with the identities $f = j_0 \circ q_0 = j_1 \circ q_1$. Since j_1 is injective, if we can show that $j_1 q_1(z) = j_0 q_0(x)$, then we will have $q_1(z) = q_1(x)$ as desired. But

$$j_1 q_1(z) \quad = \quad f(z) \quad = \quad j_0 q_0(z) \quad = \quad j_0(y) \quad = \quad j_0 q_0(x) \quad = \quad f(x) \quad = \quad j_1 q_1(x)$$

so we do have the necessary identity $q_1(z) = q_1(x)$. Therefore we have defined a mapping $H : C_0 \to C_1$ such that $H \circ q_0 = q_1$.

We now need to show that H is bijective. Suppose that $H(y) = H(y')$ and choose x, x' so that $y = q_0(x)$ and $y' = q_0(x')$. We then have

$$j_0(y) \quad = \quad j_0 q_0(x) \quad = \quad f(x) \quad = \quad j_1 q_1(x) \quad = \quad j_1 H(y)$$

$$j_0(y') \quad = \quad j_0 q_0(x') \quad = \quad f(x') \quad = \quad j_1 q_1(x') \quad = \quad j_1 H(y')$$

and since $H(y) = H(y')$ it follows that the expressions on both lines are equal, so that $j_0(y) = j_0(y')$. Since j_0 is injective, this implies $y = y'$ and hence H is injective. To show that H is surjective, express a typical element $z \in C_1$ as $q_1(x)$ for some x; then if $y = q_0(x)$ we have $z = H(y)$. This completes the proof that H is bijective.

All that remains is to show that H is unique. Suppose that $K : C_0 \to C_1$ also satisfies $K \circ q_0 = q_1$. Then if $y \in C_0$ and $y = q_0(x)$ we have

$$K(y) \quad = \quad K \circ q_0(x) \quad = \quad q_1(x) \quad = \quad H \circ q_0(x) \quad = \quad H(y)$$

and hence $K = H$, proving uniqueness.∎

IV.5 : Constructions involving functions

Exercises to work

1. Suppose that we are given an arbitrary function $g : A \to E \times F$ such that

$$g(a) \quad = \quad \big(\, u(a),\, v(a)\,\big)$$

for some functions $u : A \to E$ and $v : A \to F$. For each $(e, f) \in E \times F$ we then have $g(a) = (e, f)$ if and only if $u(a) = e$ and $v(a) = f$.

Let us apply this to the situation in the problem: Since $p_j(x) = d$ is equivalent to saying that x lies in the equivalence class d, it follows that $q(x) = (e, f)$ if and only if $x \in e$ and $x \in f$, which is equivalent to saying that $x \in e \cap f$.■

2. Once again, follow the hints for each part.

The correspondence $(B \times C)^A \longleftrightarrow B^A \times C^A$. As in the hint let p and q be the coordinate projections from $B \times C$ to B and C respectively. Given $f : A \to B \times C$, one has the associated pair $(p \circ f, q \circ f) \in B^A \times C^A$. This mapping is onto because one can use functions $u : A \to B$ and $v : A \to C$ to define a function $f(a) = (\, u(a),\, v(a)\,)$, and it is 1–1 because $pf' = pf$ and $qf' = qf$ imply that the first and second coordinates of $f(a)$ and $f'(a)$ are equal for all am so that f and f' are the same function.■

The correspondence $\left(C^B\right)^A \longleftrightarrow C^{B \times A}$. The hint outlines definitions of mappings

$$\Phi : C^{B \times A} \to \left(C^B\right)^A \qquad\qquad \Psi : \left(C^B\right)^A \to C^{B \times A}$$

that will be repeated in the argument. Given $f : B \times A \to C$, let Γ be its graph viewed as a subset of $B \times A \times C$, and for each $a \in A$ let Γ_a be given by taking the intersection

$$\Gamma \cap B \times \{a\} \times C$$

and projecting it down to $B \times C$ under the standard projection map $B \times \{a\} \times C \to B \times C$ which forgets the middle coordinate. We claim there is a unique function $g_a : B \to C$ whose graph is equal to $\Gamma(a)$; this amounts to checking that $\Gamma(a)$ is actually the graph of a function from B to C. Suppose we are given $b \in B$. Then there is a unique $c \in C$, namely $f(b, a)$, such that $(b, a, c) \in \Gamma$, and for this choice of c we also have $(b, c) \in \Gamma(a)$. Suppose now that $(b, c') \in \Gamma(a)$. Then by definition we have $(b, a, c') \in \Gamma$, which means that $c = c'$ and hence yields the required uniqueness statement. Therefore we have constructed a mapping Φ of the type described above.

To construct a map in the opposite direction, if we are given $g \in \left(C^B\right)^A$, then for each $a \in A$ we have a function $g(a) : B \to C$. Let $\Gamma(a)$ be the graph of $g(a)$, and Γ be the set of all ordered triples (b, a, c) such that (b, c) lies in $\Gamma(a)$. We claim that Γ is the graph of a function from $B \times A$ to C. Given $(b, a) \in B \times A$ we need to show there is a unique c such that $(b, a, c) \in \Gamma$. Existence follows because we can take $c = [g(a)](b)$. To see uniqueness, note that $(b, a, c') \in \Gamma$ implies $(b, c') \in \Gamma_a$, so that $c' = [g(a)](b)$. Thus we have the map Ψ as required.

Finally, to show there are 1–1 correspondences it is enough to verify that $\Psi \circ \Phi(f) = f$ for all f and $\Phi \circ \Psi(g) = g$ for all g. These follow because our constructions have the property that Γ is the set of all (b, a, c) such that $(b, c) \in \Gamma(a)$.■

IV.6 : Order types

Exercises to work

1. Define $f : [0,1) \cup (2,3) \to [0,2)$ by $f(x) = x$ if $x < 1$ and $f(x) = x - 1$ if $x \geq 2$. We claim f is strictly increasing (hence is 1–1), and we shall do this by considering several . separate cases. Suppose we have $u < v$. (1) If $v < 1$ then $f(u) = u < v = f(v)$. (2) If $u < 1 < 2 \leq v$ then $f(u) = u < 1 \leq v - 1 = f(v)$. (3) If $2 \leq u$ then $f(u) = u - 1 < v - 1 = f(v)$.

To complete the proof it is enough to show f is onto. This is straightforward: If $y < 1$ then $y = f(y)$, while if $y \geq 2$ then $f(y+1) = y$.■

2. The interval $[0,2]$ has the self-density property: If $u < v$ then there is some w such that $u < w < v$. On the other hand, $[0,1] \cup [2,3]$ does not because there is no w in this set such that $1 < w < 2$ (it lies in the reals, where such a w exists, but we are only interested in elements of the given partially ordered set here and not in any larger partially ordered set that might contain it). Since one partially ordered set has the self-density property but the other does not, they cannot have the same order type.■

3. Write $X = P(A)$, where A is an infinite set, and let $a \in A$. Then X does not have the self-density property because there is no subset $B \subset A$ strictly between $\emptyset$ and $\{a\}$.

Turning to Y, suppose we have polynomials f and g such that $f < g$. Before going further, we should stress what this means: We have $f(x) \leq g(x)$ for all real x and there is some c such that $f(c) < g(c)$; in particular, it does **NOT** mean that $f(x) < g(x)$ for all x.

In any case let $D = g - f$ so that $h > 0$, and consider $h = f + \frac{1}{2}D$. Then direct computation shows have $f \leq h \leq g$ (*i.e.*, $f(x) \leq h(x) \leq g(x)$ for all x) and $f(c) < h(c) < g(c)$, so that $f < h < g$.■

4. A positive integer d divides 28 if and only if it has the form $d = 2^a 7^b$ where a and b are integers satisfying $0 \leq a \leq 2$ and $0 \leq b \leq 1$, and likewise a positive integer e divides 45 if and only if it has the form $3 = 3^a 5^b$ where a and b are integers satsifying $0 \leq a \leq 2$ and $0 \leq b \leq 1$. If we define a mapping from $D(28)$ to $D(45)$ taking $d = 2^a 7^b$ to $e = 3^a 5^b$, then this will be the required order-isomorphism (we shall not check the details explicitly here),

On the other hand, in the notes we noted that $D(15)$ is a partially ordered set with 4 elements that is not linearly ordered, and the set $D(8)$ is the linearly ordered set consisting of 1, 2, 4 and 8 (where the divisibility ordering agrees with the usual ordering!). In particular, $D(8)$ also has four elements. However, since $D(8)$ is linearly ordered but $D(15)$ is not, it follows that these two partially ordered sets cannot have the same order type.■

5. As noted in the hint, for each $x \in \mathbf{N}$ the set of all y such that $y < x$ is finite, for it is just $\{1, 2, ..., x-1\}$. On the other hand, the set of all elements of $\mathbf{N} \times \mathbf{N}$ (with the lexicographic ordering) that precede $(1,0)$ is the infinite set of all ordered pairs of the form $(0,k)$ where k can be any nonnegative integer. Thus one of the linearly ordered sets under consideration has the property

for each element a the set elements x for which $x < a$ is finite

while the other does not, and consequently they cannot have the same order type.■

SOLUTIONS TO EXERCISES FOR

MATHEMATICS 144 — Part 5

Fall 2006

V. Number systems and set theory

V.1 : The natural numbers and integers

Exercises to work

1. Follow the hint. If we multiply out the right side of the equation $x^2+bx+c = (x-r)(x-s)$ we see that $r+s = -b$ and $rs = c$, so both these quantities must be integers. It follows that $s = -b - r$ must also be a rational number. Furthermore, by the Quadratic Formula the roots r and s are given by

$$\frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

and hence we see that $r - s = \sqrt{b^2 - 4c}$, so that the right hand side must be a rational number.

In order to proceed we need the following variant of the proof that $\sqrt{2}$ is irrational: *If a positive integer m has a rational square root, then m is a perfect square.* PROOF : We might as well assume that $m > 1$ because we know that 1 is a perfect square. Express m as a product of powers of primes

$$m \quad = \quad p_1^{r_1} \ \cdots \ p_k^{r_k}$$

and write

$$m_1 \quad = \quad p_1^{s_1} \ \cdots \ p_k^{s_k}$$

where $s_j = 0$ if r_j is even and $s_j = 1$ if r_j is odd. Then $m = m_1 m_2$ where m_2 is a perfect square (it is the product of the numbers $p_j^{r_j - s_j}$, each of which is a perfect square because the exponents are all even) and m_1 is either 1 or a product of distinct primes. Clearly $\sqrt{m}$ is rational if and only if $\sqrt{m_1}$ is rational, so it suffices to show that the latter is true if and only if $m_1 = 1$, which holds if and only if each r_j is even. Assume the contrary, and suppose that p_j is a prime dividing m_1. If $\sqrt{m_1}$ is rational then we can write it as a quotient a/b where a and b are relatively prime positive integers. We then have $m_1 b^2 = a^2$, and since p_j divides m it follows that p_j must divide a^2, which in turn means that p_j^2 must also divide a^2; by our choice of a and b it follows that p_j does not divide b. But since p_j^2 does not divide m_1, this means that p_j must divide b, contradicting the previous sentence. It follows that $m_1 = 1$ and m is a perfect square.■

By the preceding discussion, we have seen that $b^2 - 4c = d^2$ for some positive integer d. — CLAIM: If b is odd, then d is odd, and if b is even then m is even. — If b is odd, then b^2 is also odd, and hence $b^2 - 4c = d^2$ is odd, which means that m must also be odd. On the other hand, if b is even, then b^2 is divisible by 4, which means that $d^2 = b^2 - 4c$ is also divisible by 4, which in turn implies that d must be even.

We now have that

$$r \quad = \quad \frac{-b \pm d}{2}$$

where b and d are both even or both odd. In either case we know that $-b \pm d$ is even, and therefore it follows that r (and also s) must be an integer.∎

ALTERNATE APPROACH. One might also try to prove this result by saying that if p/q is a rational root of the integral polynomial $F(t)$ then q divides the term of F with maximum degree and p divides the constant term; if the coefficient of the term of maximum degree is 1, then it follows that $q = \pm 1$ and hence the rational root must be an integer. These results on rational roots of integral polynomials follow from a fundamental result of C. F. Gauss on factoring polynomials with integer coefficients, but its proof is not covered in lower division mathematics courses, so we shall include a little background here. We know that $r = p/q$ is a rational root of a rational polynomial $F(t)$ if and only if $(t - r)$ divides $F(t)$. The result of Gauss states that if we can factor an integral polynomial $A(t)$ as a product of two rational polynomials $B(t)$ and $C(t)$ of lower degree, then in fact we can factor A as a product B_1C_1, where B_1 and C_1 are integral polynomials that are rational multiples of B and C. Assuming that we have chosen p and q to have no nontrivial common factors, this means that $(qt - p)$ must divide $F(t)$ over the integers. But this means that the coefficient of the highest power of t in $F(t)$ must be divisible by q and the constant terms must be divisible by p.∎

References for the factorization result are pages 297–298 of the book by Gallian listed below and pages 162–164 of the book by Hungerford listed below:

J. A. Gallian, *Contemporary Abstract Algebra* (Fifth Ed.), Houghton-Mifflen, Boston, 2002. ISBN: 0-6188-12214-1.

T. W. Hungerford, *Algebra* (Graduate Texts in Math. Vol. 73). Springer-Verlag, New York, 1974. ISBN: 0-387-90518-9.

2. DISREGARD. [In the proof above we use the fact that the square root of an integer is rational if and only if the integer is a perfect square, so any attempt to derive the irrationality of $\sqrt{2}$ from the preceding exercise is basically circular reasoning.]

3. Follow the hint. Let B be a nonempty set of A, and let C be the set of all integers of the form $n + b$ for some $b \in B$. Since B is nonempty, so is C. Also, $b \in B \subset A$ implies $b \geq -n$, and therefore $c = n + b \in C$ implies that $c \geq 0$. By the well ordering of the nonnegative integers we know that the (nonempty) set C has a least element m, and by the construction of C we know that $m - n \in B$. We claim it is the least element of B. Given $b \in B$ we know that $b + n \in C$, and by minimality of m se know that $m \leq n - b$; subtract n from both sides to conclude that $m - n \leq b$.∎

V.2 : Finite induction and recursion

Exercises to work

1. Let $\mathbf{P}_k$ be the statement that $k^2 + 5k$ is even. Then $\mathbf{P}_0$ is true because the value of the $k^2 + 5k$ at $k = 0$ is zero, which is even. Suppose now that $\mathbf{P}_n$ is true; we then need to show that $(n+1)^2 + 5(n+1)$ is even. If we expand the latter we obtain

$$n^2 + 2n + 1 + 5n + 5 \quad = \quad (n^2 + 5n) + (2n + 6)$$

and by the induction hypothesis we know that $n^2 + 5n$ is even. However, we also know that $2n + 6$ is even, and therefore the displayed quantity is expressed as a sum of two even integers and hence must be even itself. Thus we have shown that for all n we have $\mathbf{P}_n \implies \mathbf{P}_{n+1}$, and this means that the statement in the exercise is true for all nonnegative integers k.■

2. Let $\mathbf{P}_n$ be the statement of the exercise for the nonnegative integer n. Strictly speaking there are two parts to this, one of which is to prove the formula for $1 + \cdots + n$ and the other of which is to do the same for $1^3 + \cdots + n^3$. Both statements are trivially true if $n = 0$, and we need to show that if $\mathbf{P}_n$ is true then $\mathbf{P}_{n+1}$ is also true.

We begin with the simpler formula, where we have

$$1 + \cdots + n + (n+1) \quad = \quad \frac{n^2+n}{2} + (n+1) \quad = \quad \frac{n^2+3n+2}{2} \quad = \quad \frac{(n+1)(n+2)}{2}$$

which shows that the first part of $\mathbf{P}_{n+1}$ is true. In the other case we have

$$1^3 + \cdots + n^3 + (n+1)^3 \quad = \quad \left(\frac{n^2+n}{2}\right)^2 + (n+1)^3 \quad =$$

$$\frac{(n^4+2n^3+n^2) + (4n^3+12n^2+12n+4)}{4} \quad = \quad \frac{n^4+6n^3+13n^2+12n+4)}{4} \quad =$$

$$\frac{(n^2+2n+1)(n^2+4n+4)}{4} \quad = \quad \frac{(n+1)^2(n+2)^2}{4} \quad = \quad \left(\frac{(n+1)(n+2)}{2}\right)^2$$

thus completing the derivation of $\mathbf{P}_{n+1}$ from $\mathbf{P}_n$.■

3. If $n = 1$ the formula is true because $1! = 1 = 1^1$. Suppose now that we have $n! \leq n^n$ for some $n \geq 1$; we want to prove that $(n+1)! < (n+1)^{(n+1)}$. — Since $(n+1)! = n!(n+1)$, we must have

$$(n+1)! \quad = \quad n!(n+1) \quad \leq n^n(n+1) \quad < \quad (n+1)^n(n+1) \quad = \quad (n+1)^{(n+1)}$$

as required. To be more precise, let $\mathbf{P}_n$ be the compound statement in the exercise. Then the preceding shows that $\mathbf{P}_1$ implies $\mathbf{P}_2$, and our argument shows that if $\mathbf{P}_n$ is true for $n \geq 2$ then $n! \leq n^n$ implies $(n+1)! < (n+1)^{(n+1)}$, which is the conclusion of $\mathbf{P}_{n+1}$.

4. As noted in the hint, the cases $n = 1$ and $n \geq 2$ must be handled separately. For a sequence f of length one, we simply take $H(f) = 1$, while for sequences of length $n \geq 1$ we take $H(f) = f_{n-1} + f_{n-2}$.■

5. The crucial point is to understand how much of the payment of P units goes towards principal and how much towards interest. The interest owed at time n, which is computed using the balance after the previous payment at time $n-1$, is equal to $r\,x_{n-1}$, so this means that $P - r\,x_{n-1}$ goes to the principal and therefore we have

$$x_n \quad = \quad x_{n-1} \; - \; (P - r\,x_{n-1}) \quad = \quad (1+r)x_{n-1} \; - \; P\ .■$$

Although the problem does not ask for it, we shall also derive the formula for finding the value of P such that the loan will be paid off after M equal payments of P units. One can use the recursive relation to find an explicit formula for x_n in terms of L, r and P:

$$x_n \quad = \quad \frac{P}{r} \; + \; (1+r)\left[S - \frac{L}{r}\right]$$

The condition that x_M should equal zero leads to the following expression for P in terms of L, r and M:

$$P = \frac{rL(1+r)^M}{(1+r)^{M+1}-1}$$

If one intends to use this formula to work out a specific problem in computing payments, it is important to remember that the payments are usually monthly, so M denotes the number of months and r denotes the monthly interest rate (converted from a percentage to a decimal fraction, which means dividing the monthly percentage rate by 100).∎

6. Following the hint, let $A = \{0\} \cup \sigma[\mathbf{N}]$. We need to show that $0 \in A$ and if $a \in A$ then $\sigma(a) \in A$. Then the third Peano axiom will imply that $A = \mathbf{N}$, and since A has only one element that is not the successor of anything else, the same must be true for $\mathbf{N}$.

The condition $0 \in A$ iis true by definition. If $a \in A$, then either $a = 0$ or $a = \sigma(b)$ for some $b \in \mathbf{N}$. In either case $\sigma(a) \in \sigma[\mathbf{N}] \subset A$, so this proves the second condition in the third Peano axiom.∎

V.3 : Finite sets

Exercises to work

1. We prove this by induction on $|A|$. If $|A| = 1$, then $A = \{a\}$ for some a and the result is true by assumption (2). Suppose the result is true for finite sets with n elements and that $|A| = n+1$. Let $a \in A$ and set $A_0 = A - \{a\}$; let $C_0 = C \cap A_0 \times B$, and let $C' = C \cap \{a\} \times B$. We then have $C = C_0 \cup C'$ and $C_0 \cap C' = \emptyset$. Furthermore, assumption (2) implies that $|C'| = k$ and $|C_0| = |A_0| \cdot k$. Therefore we have

$$|C| = |C_0| + |C_1| = |A_0| \cdot k + k =$$

$$(|A_0| + 1) \cdot k = |A| \cdot k$$

which completes the derivation of the inductive step.∎

IMPORTANT GENERALIZATION.

One can view an ordered pair as a sequence of length 2; with this interpretation, the conclusion of the exercise extends to sequences of arbitrary finite length as follows:

> **Informal version.** *Suppose that we are given a sequence of k choices $\mathbf{ch}_i$ such that at each step the number n_i of alternatives does not depend upon the previous choices. Then the total number of possible choice sequences is $n_1 \cdot \ldots \cdot n_k$.*
>
> **Formal version.** *Let S be a set of sequences of length k whose terms lie in some finite set A, and for each i such that $1 \leq i \leq n$ let S_i be the set of all restrictions of sequences in S to $\{1, \cdots, i\}$; set $S_0 = \emptyset$. Suppose that for each i such that $0 \leq i < n$, and each $y \in S_i$ the number $N(y)$ of sequences $x \in S_{i+1}$ restricting to y is independent of y, and denote this number by n_{i+1}. Then the number $|S|$ of sequences in S is equal to the product $n_1 \cdots n_k$.*

This principle plays an important role in the proofs of many formulas (for example, showing that the number of permutations of $\{1, \cdots, n\}$ is $n!$ and the fact that the number of subsets of $\{1, \cdots, n\}$ with exactly r elements is equal to

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} .$$

2. We can fit this example into the setting of the previous exercise with $A = B = \{1,2,3,4,5,6,7,8,9,10\}$. In this case the number $k = 5$, which is the number of integers that are odd if a is even and the number that are odd if k is odd. Therefore the total number of pairs in this case is equal to $10 \times 5 = 50$.■

3. By the theorem, there are as many Boolean subalgebras as there are partitions of $\{1,2,3,4\}$ into disjoint subsets. The standard way to count partitions is to do so in decreasing order of the sizes of the subsets. We then have the following:

- There is one partition containing one subset of 4 elements.
- There are three partitions containing one subset of 3 elements and one of 1 element.
- There are three partitions containing two subsets of 2 elements.
- There are six partitions containing one subset of 2 elements and two of 1 element.
- There are four partitions containing four subsets of 1 element.

Thus the total number of partitions is $1 + 4 + 3 + 6 + 4 = 18$. Furthermore, the number with two atomic elements is the number of partitions into two subsets, which are all those of the second and third types. Thus there are exactly seven subalgebras that have precisely two atomic elements.■

V.4 : The real numbers

Exercises to work

1. Suppose that x_0 and x_1 are the two elements of the set S and they are indexed so that $x_0 < x_1$. We claim that x_1 is the least upper bound of S and x_0 is the greatest lower bound of S. The fiact that they are upper bounds follows because $y \in S$ implies $x_0 \leq y \leq x_1$. Suppose that U is another upper bound for S. Then $x_1 \in S$ implies that $x_1 \leq U$, which is precisely the condition for x_1 to be the least upper bound. Similarly, if L is a lower bound for S, then $L \leq x_0$, which is precisely the condition for x_0 to be the greatest lower bound for S.■

2. The least upper bound of $A \cup B$ is the larger of u and v. To prove this, let w be the larger of u and v. Then $x \in A \cup B$ implies $x \in A$ or $x \in B$, which in turn implies $x \leq u$ or $x \leq v$. In either case we have $x \leq w$, so w is an upper bound for $A \cup B$.

To see it is the least upper bound for $A \cup B$, suppose we have $z < x$; we need to show that z cannot be an upper bound for the union. Suppose that $w = u$. Then by the definition of least upper bound we know that there is some $a \in A$ such that $a > z$. Since z is not an upper bound for A it cannot be an upper bound for the larger set $A \cup B$. Likewise, if $z = v$ then there is some $b \in B$ such that $b > z$. Since z is not an upper bound for B it cannot be an upper bound for the larger set $A \cup B$. Therefore in either case we know that z cannot be an upper bound for $A \cup B$, and hence w must be a least upper bound for $A \cup B$.■

3. First of all, 0 is greater than every negative number, so 0 is an upper bound for A. Suppose now that $a < 0$. Then a cannot be an upper bound for A because we have $a < \frac{1}{2}a < 0$; thus if U is an upper bound for A then $U \geq 0$, and hence 0 must be the least upper bound for A.■

4. Since x is the least upper bound for A, we know that for each positive integer n the number $a - \frac{1}{n}$ is not an upper bound, and hence there is some $a_n \in A$ such that $a - \frac{1}{n} < a_n < a$. We claim $\lim_{n\to\infty} a_n = a$. Let $\varepsilon > 0$, and choose N such that $n \geq N$ implies $\frac{1}{n} < \varepsilon$. Then $n \geq N$ implies

$$a \;>\; a_n \;>\; a - \frac{1}{n} \;\geq a - \frac{1}{N} \;>\; a - \varepsilon$$

so that $|a_n - a| < \varepsilon$ as required.■

V.5 : Familiar properties of the real numbers

Exercises to work

1. There are many ways of doing this problem. For example, we can start by saying that there is a rational number r_0 such that $a < r_0 < b$ and another rational number r_1 such that $r_0 < r_1 < b$. An entire sequence of numbers r_n for $n > 1$ such that $r_n < \cdots < r_2 < r_1$ may be defined by setting

$$r_n \;=\; r_0 \;+\; \frac{r_1 - r_0}{n}$$

or alternatively one can take a sequence such that $a < r_0 < r_1 < r_2 < \cdots < r_n < \cdots < b$.■

2. Each case will be handled separately. It is probably worthwhile to begin by observing that we can write 1 in "base 16 decimal-like" notation as 0.FFFFFFF...$_{\mathbf{HEX}}$, because we have the following geometric series identity which works for all $n > 1$:

$$\sum_{k=1}^{\infty} \frac{n-1}{n} \cdot \left(\frac{1}{n}\right)^k \;=\; \frac{n-1}{n} \cdot \frac{1}{1 - (1/n)} \;=\; 1$$

In the discussion below we shall always denote hexadecimal expansions by appending the subscript "$_{\mathbf{HEX}}$" as above; for example, $14_{\mathbf{HEX}}$ is equal to 20 (in base 10).

The easy cases. If k divides 16 evenly, then just as for decimals the expansion is given by $16/k$ in the first position and zeros afterwards, or equivalently by $(16/k) - 1$ in the first position and F's afterwards. Thus we have that $\frac{1}{2} = 0.800000...{}_{\mathbf{HEX}}$, $\frac{1}{4} = 0.400000...{}_{\mathbf{HEX}}$, and $\frac{1}{8} = 0.200000...{}_{\mathbf{HEX}}$.

The case $\frac{1}{3}$. The algorithm tells us exactly how to proceed. Start with $16 = x_1 \cdot 3 + y_1$, $16y_1 = x_2 \cdot 3 + y_2$, and so forth, obtaining $16 = 5 \cdot 3 + 1$, $16 = 16 \cdot 1 = 5 \cdot 3 + 1$, and similarly for every other value. The terms in the expansion are the x_j's, so this means that $\frac{1}{3} = 0.5555555555...{}_{\mathbf{HEX}}$.

The case $\frac{1}{5}$. In this case the algorithm yields $16 = 3 \cdot 5 + 1$, $16 = 16 \cdot 1 = 3 \cdot 5 + 1$, and similarly for every other value, so this means that $\frac{1}{5} = 0.3333333333...{}_{\mathbf{HEX}}$.

The case $\frac{1}{6}$. In this case the algorithm yields $16 = 2 \cdot 6 + 4$, $16 \cdot 4 = 64 = 10 \cdot 6 + 1$, and similarly for every other value, so this means that $\frac{1}{6} = 0.2\text{AAAAAAAAA}...{}_{\mathbf{HEX}}$.

The case $\frac{1}{7}$. In this case the algorithm yields $16 = 2 \cdot 7 + 2$, $32 = 16 \cdot 2 = 4 \cdot 7 + 4$, $64 = 16 \cdot 4 = 9 \cdot 7 + 1$, $16 = 2 \cdot 7 + 2$, and one has a periodic pattern of length 3 for the remaining values, so this means that $\frac{1}{7} = 0.249249249249...{}_{\mathbf{HEX}}$.

The case $\frac{1}{9}$. In this case the algorithm yields $16 = 1 \cdot 9 + 7$, $112 = 16 \cdot 7 = 12 \cdot 9 + 4$, $64 = 16 \cdot 4 = 7 \cdot 9 + 1$, $16 = 1 \cdot 9 + 7$, and one has a periodic pattern of length 3 for the remaining values, so this means that $\frac{1}{9} = 0.1C71C71C71C7..._{\textbf{HEX}}$.

The case $\frac{1}{10}$. In this case the algorithm yields $16 = 1 \cdot 10 + 6$, $16 \cdot 6 = 96 = 6 \cdot 10 + 6$, and similarly for every other value, so this means that $\frac{1}{10} = 0.1999999999..._{\textbf{HEX}}$.

This completes the list of examples in the exercise, but of course one could continue to find hexadecimal expansions for all of the fractions $\frac{1}{k}$.■

3. The point of this exercise is that x has an eventually periodic decimal expansion if and only if $f(x)$ does.

Suppose that x is rational and that it has a decimal expansion that is eventually periodic with period p; in other words, there is some N such that for each $n \geq N$ the decimal digits x_n for x satisfy $x_n = x_{n+p}$. What can one say about the decimal digits y_n for $y = f(x)$ if $n \geq 2N$? If n is even then $y_n = 0$ and thus we trivially have $y_{n+2p} = 0 = y_n$, while if n is an odd number of the form $2m - 1$ then $2m - 1 \geq 2N$ implies $m \geq N$, so that $y_{2m-1} = x_m = x_{m+p} = y_{2p+2m-1}$. Thus the decimal expansion of $y = f(x)$ is eventually periodic, so that $f(x)$ is rational if x is rational.

Suppose now that $f(x)$ is rational. Since we know that $f(0) = 0$, we need only consider the case where $f(x)$ and x are both nonzero. — The conclusion is also trivial if $f(x)$ is a finite decimal fraction (in which case the same is true for x), so let us also assume that there are infinitely many decimal digits that are nonzero for x. Since only the odd entries are nonzero, it follows that the period of the tail end of the expansion must be even (this uses the fact that there are infinitely many nonzero terms so that there is a nonzero entry in the repeating part of the decimal exapansion). Thus if we let $y = f(x)$ as before, then we have some $2N$ and p such that $m \geq N$ implies $y_{2p+2m-1} = y_{2m-1}$. Thus for all m sufficiently large we also have $x_{m+p} = x_m$ as well.■

4. Follow the hints as usual. We want to apply the summation formulas

$$\sum_{i,j \geq 1} a_{i,j} \quad = \quad \sum_{i \geq 1} \sum_{j \geq 1} a_{i,j} \quad = \quad \sum_{j \geq 1} \sum_{i \geq 1} a_{i,j}$$

where $a_{i,j} = 2^{1-(i+j)}$ if $i \leq j$ and 0 otherwise. If we sum first over j holding i fixed and then sum over i, we find that the sum of this series is equal to the Swineshead series

$$\sum_{k \geq 1} \frac{k}{2^k}$$

as indicated in the problem. What happens if we sum over i holding j fixed and then sum over j? We obtain

$$\sum_{j \geq 1} \sum_{i \geq 1} 2^{1-(i+j)} \quad = \quad \sum_{j \geq 1} 2^{2-j} \quad = \quad 2$$

which is the value that Swineshead and Oresme computed in the 14th century.■

SOLUTIONS TO EXERCISES FOR

MATHEMATICS 144 — Part 6

Fall 2006

VI. Infinite constructions in set theory

VI.1 : Indexed families and set - theoretic operations

Exercises to work

1. We first verify the statement about unions. Suppose that $x \in \cup_{j \in J} A_j$, and choose $j(0) \in J$ such that $x \in A_{j(0)}$. Then the inclusion hypothesis implies that $x \in C_{j(0)}$, which in turn implies that $x \in \cup_{j \in J} C_j$. Therefore $\cup_{j \in J} A_j$ is a subset of $\cup_{j \in J} C_j$.

We now verify the statement about intersections. Suppose that $x \in \cap_{j \in J} A_j$, so that $x \in A_j$ for every $j \in J$. The inclusion hypothesis now implies that $x \in C_j$ for every $j \in J$, and therefore we must have $x \in \cap_{j \in J} C_j$. Therefore $\cap_{j \in J} A_j$ is a subset of $\cap_{j \in J} C_j$.■

2. We prove the assertions in order. Suppose that $x \in S - \cup_{j \in J} A_j$. Then $x \notin \cup_{j \in J} A_j$, or equivalently there is no j such that $x \in A_j$. Therefore we have $x \notin A_j$ for all j, and since $x \in S$ this means $x \in S - A_j$ for all j. The latter in turn implies that $x \in \cap_{j \in J} S - A_j$, and therefore we have $S - \cup_{j \in J} A_j \quad \subset \quad \cap_{j \in J} S - A_j$.

Conversely, if $x \in \cap_{j \in J} S - A_j$, then $x \notin A_j$ for each j, so that there is no j satisfying $x \in A_j$ and hence $x \notin \cup_{j \in J} A_j$. Since $x \in S$, it follows that $x \in S - \cup_{j \in J} A_j$, and this plus the conclusion of the previous paragraph establishes one of the De Morgan laws.

We now turn to the other De Morgan law. Suppose that $x \in S - \cap_{j \in J} A_j$. Then there is some $j(0) \in J$ such that $x \notin A_{j(0)}$, and accordingly we have $x \in S - A_{j(0)}$. Now the latter set is a subset of the union $\cup_{j \in J} S - A_j$ by the definition of this union, and therefore we have $S - (\cap_{j \in J} A_j) \ \subset \ \cup_{j \in J} S - A_j$.

Conversely, if $x \in \cup_{j \in J} S - A_j$, then there is some $j(0)$ such that $x \in S - A_{j(0)}$, so that $x \notin A_{j(0)}$. The last statement implies that $x \notin \cap_{j \in J} A_j$, and since $x \in S$ it follows that $x \in S - (\cap_{j \in J} A_j)$. As in the discussion of the first De Morgan law, this plus the conclusion of the previous paragraph establishes the second De Morgan law.■

3. (a) Suppose first that $x \in (\cup_i A_i) \cap (\cup_j B_j)$. Then one can find indices $i(0)$ and $j(0)$ such that $x \in A_{i(0)}$ and $x \in B_{j(0)}$, and hence $x \in A_{i(0)} \cap B_{j(0)}$, so that $x \in \cap_{i,j} \ (A_i \ \cup \ B_j)$. — Conversely, if x lies in the latter set, then one can find indices $i(0)$ and $j(0)$ such that $x \in A_{i(0)} \cap B_{j(0)}$. Since $A_{i(0)}$ is a subset of $\cup_i A_i$ and $B_{j(0)}$ is a subset of $\cup_j B_j$, it follows that the intersection $A_{i(0)} \cap B_{j(0)}$ is a subset of $(\cup_i A_i) \ \cap \ (\cup_j B_j)$. This proves the first identity in (a).■

We now turn to the second identity. Suppose that $x \in (\cap_i A_i) \ \cup \ (\cap_j B_j)$. Then either $x \in \cap_i A_i$ or $x \in \cap_j B_j$. In the first case we have $x \in A_i$ for all i and in the second we have $x \in B_j$ for all j. Therefore in both cases we have $x \in A_i \cup B_j$ for all i and j., so that $x \in \cap_{i,j} \ (A_i \ \cup \ B_j)$. — Conversely, if x lies in the latter set, then for each ordered pair (i, j) we either have $x \in A_i$ or

$x \in B_j$. It suffices to show that if $x \notin \cap_i\ A_i$, then we must have $x \in \cap_j\ B_j$. However, if x does not belong to the first intersection, then for some $i(0)$ we have $x \notin A_{i(0)}$, and thus for all ordered pairs $(\,i(0), j\,)$ we must have $x \in B_j$. The last statement implies that $x \in \cap_j\ B_j$, which is what we needed to verify.■

(b) Suppose that $x \in A_k$ for some k, and choose j such that $k \in I_j$. We then have $x \in \cup\ \{A_i \mid i \in I_j\}$, which in turn implies

$$x \;\in\; \bigcup_{j \in J} \left(\bigcup_{i \in I_j} A_i \right) .$$

Conversely, if x belongs to the latter set, then for some j we have $x \in \cup\ \{A_i \mid i \in I_j\}$, which in turn means that $x \in A_i$ for some i, so that $x \in \cup_k\ A_k$. This proves the first identity in (b).■

We now turn to the second identity. Suppose that $x \in A_k$ for all k. Then for each j we have $x \in \cap\ \{A_i \mid i \in I_j\}$, and hence we also have

$$x \;\in\; \bigcap_{j \in J} \left(\bigcap_{i \in I_j} A_i \right) .$$

Conversely, if x belongs to the latter set, then for all j we have $x \in \cap\ \{A_i \mid i \in I_j\}$, which in turn means that $x \in A_i$ for all i, so that $x \in \cap_k\ A_k$. This proves the second identity in (b).■

4. (a) Suppose first that $(x, y) \in (\cup_i\ A_i)\ \times\ (\cup_j\ B_j)$. Then one can find some indices $i(0)$ and $j(0)$ such that $x \in A_{i(0)}$ and $y \in B_{j(0)}$. Therefore we have $(x, y) \in A_{i(0)} \times B_{j(0)}$. Since the latter is contained in $\cup_{i,j}\ (A_i \times B_j)$ it follows that $(x, y) \in \cup_{i,j}\ (A_i \times B_j)$. — Conversely, it (x, y) belongs to the latter set, then one can find some indices $i(0)$ and $j(0)$ such that $(x, y) \in A_{i(0)} \times B_{j(0)}$, and therefore it follows that (x, y) belongs to $(\cap_i\ A_i)\ \times\ (\cap_j\ B_j)$. This proves the first identity in (a).■

We now turn to the second identity. Suppose that $(x, y) \in (\cap_i\ A_i)\ \times\ (\cap_j\ B_j)$. Then for all i and j we have $x \in A_i$ and $y \in B_j$, so that $x \in A_i \times B_j$ for all i and j, and hence we have $(x, y)\ \in\ \cap_{i,j}\ (A_i \times B_j)$. — Conversely, if (x, y) belongs to the latter set, then for each i and j we know that $x \in A_i$ and $y \in B_j$, so that $(x, y) \in (\cap_i\ A_i)\ \times\ (\cap_j\ B_j)$. This proves the second identity in (a).■

(b) First of all, we need to show that for each j we have $\cap_i\ X_i \subset X_j \subset \cup_i\ X_i$. If y lies in the intersection on the left hand side, then it lies in each X_i and in particular it lies in X_j, so the first inclusion is true. Likewise, if $y \in X_j$ then trivially we have $f \in X_i$ for some i and hence $y \in \cup_i\ X_i$.

To complete the second part of the problem, we need to show that if the sets U and V satisfy

$$U \;\subset\; X_j \;\subset\; V$$

for every j, then we have $U \subset \cap_i\ X_i$ and $\cup_i\ X_i \subset V$. — If $y \in U$, then by hypothesis we have $y \in X_i$ for all i and since the intersection is defined by this condition we have $U \subset \cap_i\ X_i$. Also, if $y \in \cup_i\ X_i$, then for some j we have $y \in X_j$. By assumption $X_j \subset V$, and therefore we also have $y \in V$. But this means that every element of $\cup_i\ X_i$ is also in V, so that $X_j \subset V$ as required.■

VI.2 : Infinite Cartesian products

Exercises to work

1. The main idea is to apply the Universal Mapping Property:

Let $\{A_\alpha \mid \alpha \in \Lambda\}$ be a family of nonempty sets, and suppose that we are given data consisting of a set P and functions $h_\alpha : P \to A_\alpha$ such that for EVERY *collection of data $(S, \{f_\alpha : S \to A_\alpha\})$ there is a unique function $f : S \to P$ such that $h_\alpha \circ f = f_\alpha$ for all α. Then there is a unique $1-1$ correspondence $\Phi : \prod_\alpha A_\alpha \to P$ such that $h_\alpha \circ \Phi$ is projection from $\prod_\alpha X_\alpha$ onto A_α for all α.*

Application to the exercise. For each k let P_k denote the product of objects whose index belongs to J_k and denote its coordinate projections by p_i. The conclusions amount to saying that there is a canonical morphism from $\prod_k P_k$ to $\prod_i X_i$ that has an inverse morphism. Suppose that we are given morphisms f_i from the same set S to the various sets X_i. If we gather together all the morphisms for indices i lying in a fixed subset J_k, then we obtain a unique map $g_k : S \to P_k$ such that $p_i \circ g_k = f_i$ for all $i \in J_k$.

Let $q_k : \prod_\ell P_\ell \to P_k$ be the coordinate projection. Taking the maps g_k that have been constructed, one obtains a unique map $F : S \to \prod_k P_k$ such that $q_k \circ F = g_k$ for all k. By construction we have that $p_i \circ q_k \circ F = f_i$ for all i. If there is a unique map with this property, then $\prod_k P_k$ will be isomorphic to $\prod_i X_i$ by the Universal Mapping Property. But suppose that θ is any map with this property. Once again fix k. Then $p_i \circ q_k \circ F = p_i \circ q_k \circ \theta = f_i$ for all $i \in J_k$ implies that $q_k \circ F = q_k \circ \theta$, and since the latter holds for all k it follows that $F = \theta$ as required.∎

2. Once again we use the Universal Mapping Property. If we are given a sequence of set-theoretic functions $f_i : X_i \to Y_i$, then we obtain a corresponding set of functions $f_i^* : \prod_j X_j \to Y_i$ defined by the identities

$$f_i^* \quad = \quad f_i \circ p_i^X$$

where $p_i^x : \prod_j X_j \to X_i$ is projection. Thus the Universal Mapping Property yields a unique mapping

$$F = \prod_i f_i \;:\; \prod_i X_i \quad \to \quad \prod_i Y_i$$

such that $p_i^Y \circ F = f_i^* = f_i \circ p_i^X$ for each i, where π_i^X and π_i^Y denote the i^{th} coordinate projections for $\prod_i X_i$ and $\prod_i Y_i$ respectively.

The assertion that F is an identity mapping if each f_i is an identity mapping follows because from the uniqueness part of the Universal Mapping Property, for the identity mapping on the product satisfies the displayed equation if each of the mappings f_i is an identity mapping.

Finally, the assertion about composites can be verified as follows: Let $H = \prod_j g_j \circ f_j$. Then for each i we have

$$p_i^Z \circ H \quad = \quad g_i \circ f_i \circ \pi_i^X \quad = \quad g_i \circ p_i^Y \circ F \quad = \quad p_i^Z \circ G \circ F$$

and therefore $H = G \circ F$ by the Universal Mapping Property.∎

3. Follow the hint. Since each f_j is a bijection we have inverse mappings $g_j = f_j^{-1}$. By the preceding exercise we then have

$$\prod_j f_j \circ \prod_j g_j \quad = \quad \prod_j (f_j \circ g_j) \quad = \quad \prod_j \mathrm{id}(Y_j) \quad = \quad \mathrm{id}\left(\prod_j Y_j\right)$$

and we also have

$$\prod_j g_j \circ \prod_j f_j \quad = \quad \prod_j (g_j \circ f_j) \quad = \quad \prod_j \mathrm{id}(X_j) \quad = \quad \mathrm{id}\left(\prod_j X_j\right)$$

so that the product of the inverses $\prod_j g_j$ is an inverse to $\prod_j f_j$.■

4. Both statements are **TRUE.** — To prove the first one, let $\mathbf{u}, \mathbf{v} \in \prod_j X_j$ with coordinates u_j and v_j respectively, and suppose that $\prod_j f_j(\mathbf{u}) = \prod_j f_j(\mathbf{v})$. This means that the j^{th} coordinates are equal for every j. But the j^{th} coordinates of the given elements are $f_j(u_j)$ and $f_j(v_j)$ respectively. Since each f_j is 1–1 it follows that $u_j = v_j$ for all j, which in turn means that $\mathbf{u} = \mathbf{v}$. Therefore $\prod_j f_j$ is 1–1 if each f_j is 1–1.

Suppose now that each map f_j is onto, and let $\mathbf{y} \in \prod_j Y_j$ with coordinates y_j. Since each f_j is onto for each j there is an element $x_j \in X_j$ such that $f_j(x_j) = y_j$. If we take $\mathbf{x} \in \prod_j X_j$ such that the j^{th} coordinate is x_j for each j, then it follows that $\prod_j f_j(\mathbf{x}) = \mathbf{y}$ and hence $\prod_j f_j$ is onto.■

5. We shall use the following result from Unit IV: *Let X and Y be sets, let $\varphi : X \to Y$ be a function, let R be a binary relation on X, and let $\mathbf{E}$ be the equivalence relation generated by R. Suppose that for all $u, v \in X$ we know that $u\,R\,v$ implies $\varphi(u) = \varphi(v)$. Then for all $x, y \in X$ such that $x\,\mathbf{E}\,y$ we have $\varphi(x) = \varphi(y)$.*

To solve the problem, let R be the binary relation on B such that uBv if and only if there is some $x \in A$ such that $u = f(x)$ and $v = g(x)$, let $\mathbf{E}$ be the equivalence relation generated by R, let C be the corresponding set of equivalence classes, and let $p : B \to C$ be the equivalence class projection. By construction we have $p \circ f(x) = p \circ g(x)$ for all $x \in A$.

Suppose now that we have a function $q : B \to D$ such that $q \circ f = q \circ g$. We need to define a function $h : C \to D$ such that h sends the equivalence class $[b]$ of b to $q(b)$. The main problem is to verify that h is well defined; *i.e.*, it does not depend upon the choice of an element of b representing a given equivalence class. If we can show that h is well defined, the it will follow that $h \circ p = q$; furthermore if we also have $k \circ p = q$, then for each $c \in C$ we may write $c = p(b)$ for some b and hence

$$k(c) \quad = \quad k \circ p(b) \quad = \quad q(b) \quad = \quad h \circ p(b) \quad = \quad h(x)$$

so that $h = k$. — By the proposition quoted in the first paragraph, it suffices to show that if $u\,R\,v$ then $q(u) = q(v)$, and by definition of R this reduces to showing that for each $x \in A$ we have $q \circ f(x) = q \circ g(x)$; this equation holds by our hypothesis on q, and therefore by the proposition we know that h is well defined. As noted before, this completes the proof.■

6. First of all, we observe a consequence of the uniqueness statement. Namely, the only maps $\varphi : C \to C$ and $\psi : E \to E$ such that $p \circ \varphi = p$ and $q \circ \psi = q$ are the identities on C and E respectively.

By the universal mapping properties for coequalizers, there are unique maps $H : C \to E$ such that $r = H \circ p$ and $K : E \to C$ such that $p = K \circ r$. It follows that $r = K \circ H \circ r$ and $p = H \circ K \circ p$, and therefore by the first sentence we conclude that $H \circ K$ and $K \circ H$ are the identity mappings on C and E respectively. As usual, this implies that both H and K are bijections.■

VI.3 : Transfinite cardinal numbers

Exercises to work

1. Let S be the set in question, and let $S[n]$ denote the set of subsets with n elements. It will suffice to show that each $A[n]$ is countable, because a countable union of countable sets is countable. In fact, since there is only one subset with no elements, we might as well assume that $n \geq 1$.

Since S is countable it has a well-ordering. Define a map F from $S[n]$ to $\Pi^n S$ — the product of n copies of S with itself — such that the coordinates of $F(B)$ are the elements of B in order; *i.e.*, the first coordinate is the least element b_0, the second is the least element of those remaining after b_0 is removed, and so on. This defines a 1–1 mapping into $\Pi^n S$, which is a countable set. Hence each set $S[n]$ is countable as required.

To see the final assertion, note that if S is finite then the set $P(S)$ of all subsets is finite, while if S is infinite, then the set $\mathbf{F}_1(S)$ of subsets with exactly one element is in 1–1 coorespondence with S, and hence both $\mathbf{F}_1(S)$ and $P(S) \supset \mathbf{F}_1(S)$ are infinite.■

2. The equivalence class projection from S to S/E is an onto mapping, and since S is countable the results of Section 3 imply that S/E must also be countable.■

VI.4 : Countable and uncountable sets

Exercises to work

1. Let A, B, C, D be sets such that $|A| = \alpha$, $|B| = \beta$, $|C| = \gamma$, and $|D| = \delta$. The cardinal number inequalities imply the existence of 1–1 mappings $f : A \to B$ and $g : C \to D$. These maps in turn define mappings $f \amalg g : A \amalg C \to B \amalg D$ and $f \times g : A \times C \to B \times D$ as follows:

$$[f \amalg g]\,(a,1) \quad = \quad (\,f(a), 1\,)$$

$$[f \amalg g]\,(c,2) \quad = \quad (\,g(c), 2\,)$$

$$[f \times g]\,(c,b) \quad = \quad (\,f(a),\, g(c)\,)$$

Since $\alpha + \gamma = |A \amalg C|$ and $\alpha \cdot \gamma = |A \times C|$ and similarly if β, δ, B, D replace α, γ, A, C the conclusion of the problem will follow if we can verify that $f \amalg g$ and $f \times g$ are both 1–1.

To see that $f \amalg g$ is 1–1, suppose that we have two classes (u, i) and (v, j) which have the same image under this map. By definition of $f \amalg g$ we know that the second coordinates satisfy $i = j$ so that either this second coordinate is 1 and $u, v \in A$ or else this second coordinate is 2 and $u, v \in B$. In each case the injectivity of f and g imply that the images of (u, i) and (v, i) are the same if and only if $u = v$. Therefore $f \amalg g$ is injective if f and g are.

To see that $f \times g$ is 1–1, suppose that we have $f \times b(a, c) = f \times g(a', c')$. By definition of $f \times g$ we conclude that

$$\big(\,f(a),\, g(c)\,\big) \quad = \quad \big(\,f(a'),\, g(c')\,\big)$$

and since ordered pairs are determined by their coordinates the latter implies that $f(a) = f(a')$ and $g(c) = g(c')$. Since f and g are injective this implies that $a = a'$ and $c = c'$ so that $(a, c) = (a', c')$ and hence $f \times g$ is injective if f and g are.■

2. Choose A so that $|A| = \alpha$. Then $A \times \emptyset = \emptyset$ implies that $\alpha \cdot 0 = 0$. Also α^1 is the cardinality of the set of all functions from $\{1\}$ to A, which is in 1–1 correspondence with A under the mapping which sends $f : \{1\} \to A$ to the value $f(1) \in A$. Therefore we have $\alpha^1 = |A| = \alpha$. Finally, 1^α is the cardinality of the set of all functions from A to $\{1\}$, and since there is a unique function of this type (the function whose value at every element of A is equal to 1), it follows that $1^\alpha = 1$.■

3. We shall split the proof into several steps.

(i) Suppose that $\alpha = \aleph_0$ or $2^{\aleph_0}$. Prove that $\alpha^\alpha = 2^\alpha$. Suppose that $\alpha = \aleph_0$ and $\beta = 2^\alpha$. Then $\beta^\alpha = 2^\alpha$.

(ii) Let X be a set, and let $\Sigma(X)$ denote the set of bijections from X to itself. Suppose that $\varphi : X \to Y$ is a bijection of sets. Prove that there is a bijection $\varphi_* : \Sigma(X) \to \Sigma(Y)$ such that $\varphi_*(h) = \varphi \circ h \circ \varphi^{-1}$ for all $h \in \Sigma(X)$.

(iii) Suppose that $|X| = \alpha$, where $\alpha = \aleph_0$ or $2^{\aleph_0}$. Prove that $|\Sigma(X)| = \alpha^\alpha = 2^\alpha$.

Proof of (i). We shall prove the statements in the individual sentences separately. For the first sentence, we have $2^\alpha \leq \alpha^\alpha$; since $\alpha \cdot \alpha = \alpha$ for these cardinal numbers we also have

$$\alpha^\alpha \quad \leq \quad (2^\alpha)^\alpha \quad = \quad 2^{\alpha \cdot \alpha} \quad = \quad 2^\alpha$$

and hence the Schröder-Bernstein Theorem implies that the left and right hand sides are equal.■

IMPORTANT GENERALIZATION. This argument works for an infinite cardinal number α if we know that $\alpha \cdot \alpha = \alpha$. By the results of Section VII.4 this equation holds for all infinite cardinal numbers, and therefore it follows that **the conclusion of Part** (i) **is true for all infinite cardinal numbers.**■

Proof of (ii). It follows immediately that the construction $\Phi_*(f) = \varphi \circ f \circ \varphi^{-1}$ defines a mapping of function sets from $\mathbf{F}(X, X)$ to $\mathbf{F}(Y, Y)$. We need to show that it sends the subset $\Sigma(X)$ to $\Sigma(Y)$. In other words, if f is a bijection we need to check that $\varphi \circ f \circ \varphi^{-1}$ is also a bijection. But this follows immediately because the latter is a composite of bijections and a composite of bijections is also a bijection.

Let $\psi : Y \to S$ be the inverse of φ. Then by the same reasoning as above we have a map $\psi_* : \Sigma(Y) \to \Sigma(X)$, and it will suffice to show that the composites $\psi_* \circ \varphi_*$ and $\varphi_* \circ \psi_*$ are both identity mappings. Consider the following chains of equations:

$$\psi_* \circ \varphi_*(f) \quad = \quad \psi \circ \varphi \circ f \circ \varphi^{-1} \circ \psi^{-1} \quad = \quad \psi \circ \varphi \circ f \circ \psi \circ \varphi \quad = \quad \mathrm{id}_X \circ f \circ \mathrm{id}_X \quad = \quad f \quad = \quad \mathrm{Identity}(f)$$

$$\varphi_* \circ \psi_*(g) \quad = \quad \varphi \circ \psi \circ g \circ \psi^{-1} \circ \varphi^{-1} \quad = \quad \varphi \circ \psi \circ g \circ \varphi \circ \psi \quad = \quad \mathrm{id}_Y \circ g \circ \mathrm{id}_Y \quad = \quad g \quad = \quad \mathrm{Identity}(g)$$

It follows that φ_* is bijective and ψ_* is its inverse.■

Proof of (iii). Assume that X is either $\mathbf{N}$ or $\mathbf{R}$. Since $\Sigma(X) \subset \mathbf{F}(X, X)$ by definition, it follows that if $|X| = \alpha$ then $|\Sigma(X)| \leq \alpha^\alpha$. By the Schröder-Bernstein Theorem it will suffice to prove the reverse inequality.

Define a map $\sigma : \mathbf{F}(X, X) \to \Sigma(X \times X)$ so that for each $u : X \to X$ we have the following description of $\sigma_u \in \Sigma(X \times X)$:

[1] $\sigma_u(y, 0) = (\, y,\, u(y)\,)$

[2] $\sigma_u(\, y,\, u(y)\,) = (y, 0)$

[3] $\sigma_u(y, z) = (y, z)$ otherwise.

In words, σ_u interchanges elements of the form $(y, 0)$ with elements of the form $(\, y.\, u(y)\,)$ and leaves everything else fixed.— Note that if $X = \mathbf{R}$ the mapping σ_u is almost never going to be continuous. By construction each σ_u defines a mapping from X to itself, and in fact the composites $\sigma_u \circ \sigma_u$ are all equal to the identity map on $X \times X$. This shows that each σ_u is a bijection, and in fact each such map is equal to its own inverse.

We now need to show that σ is an injection. However, if $\sigma_u = \sigma_v$ then for every $y \in X$ we have $\sigma_u(y, 0) = \sigma_v(y, 0)$, which implies that $u(y) = v(y)$; it follows that if $\sigma_u = \sigma_v$, then $u = v$ as required.

The preceding argument shows that $\alpha^\alpha \leq |\Sigma(X \times X)|$. Since $\alpha \cdot \alpha = \alpha$ for the sets X we are considering, we may now apply (i) to conclude that $|\Sigma(X \times X)| = |\Sigma(X)|$, and therefore we also have $\alpha^\alpha \leq |\Sigma(X)|$. As noted previously in this exercise, we may now conclude that equality actually holds in the latter expression. Finally, we may now apply Part (i) to see that $2^\alpha = |\Sigma(X)|$ also holds.■

Determination of $|\Sigma(X)|$ for an arbitrary set X. More generally, it is possible to describe $|\Sigma(X)|$ as a function of $|X|$ in a very straightforward manner. Not surprisingly, the finite and transfinite cases must be handled separately.

The finite case. If X is finite and $|X| = n > 0$, then there is a 1–1 correspondence between $\Sigma(X)$ and the symmetric group Σ_n of permutations of $\{1, 2, \cdot, n\}$. It is well known that Σ_n contains $n!$ elements. Further information on this may be found in Section 4.3 of Rosen, and particularly on page 321.

The transfinite — or infinite — case. The proof of the preceding exercise is valid for all infinite sets X whose cardinal numbers satisfy $|X \times X| = |X|$. At one point in the argument we defined a map using $0 \in \mathbf{R}$, but in general one can carry out the construction replacing 0 by some arbitrary fixed element $x_0 \in X$. As noted above, by the results of Section VII.4 the identity $|X \times X| = |X|$ holds for all infinite sets, and consequently the proof implies that for every infinite set X we have $|\Sigma(X)| = 2^{|X|} = |X|^{|X|}$.

Relations between the finite and transfinite cases. There is a loose connection between the computations for $|\Sigma(X)|$ in the finite and transfinite cases ($n!$ versus α^α) in terms of a classical asymptotic formula for estimating $n!$ discovered by A. de Moivre (1667–1754) and J. Stirling (1692–1770), which is usually known as **Stirling's Formula**:

$$\lim_{n \to \infty} \frac{n!}{n^n \sqrt{2\pi n}\, e^{-n}} \quad = \quad 1$$

A discussion of this formula and a relatively elementary derivation of it may be found at the following online site:

`http://en.wikipedia.org/wiki/Stirling's_formula`

The formula implies that for all large values of n the percentage error for estimating $n!$ by the denominator goes to 0 as $n \to \infty$. However, for several reasons it would be stretching things too far to assert that the results $|\Sigma_n| = n!$ and $|\Sigma(\mathbf{N})| = |\mathbf{N}|^{|\mathbf{N}|}$ somehow "fit together continuously" in some precise matehamtical sense.■

4. If $\mathbf{c} = |\mathbf{R}|$ then we have

$$\mathbf{c} \quad \leq \aleph_0 \cdot \mathbf{c} \quad \leq \quad \mathbf{c} \cdot \mathbf{c} \quad = \quad \mathbf{c}$$

and therefore it is enough to show that for each nonnegative integer n the set of all subsets with n elements has cardinality $\mathbf{c}$, and likewise for the set of all countably infinite subsets.

Let n be a positive integer. As in Exercise VI.3.1, define a map from subsets of $\mathbf{R}$ with n elements into $\mathbf{R}^n$ such that the first coordinate is the least element of the set, the second coordinate is the smallest of the remaining elements, and so on. We can always find least elements for finite subsets because the real numbers are linearly ordered. This shows that the cardinality of the set of subsets with n elements is less than or equal to the cardinality of $\mathbf{R}^n$, which is $\mathbf{c}$. On the other hand, given a real number r_0, it is easy to find a subset with n elements whose least element is r_0, so this gives a 1–1 mapping from $\mathbf{R}$ into the set of all such subsets (specifically, let the second element be $r_0 + 1$, *etc.*). Therefore the cardinality of the set of all subsets with exactly n elements is $\mathbf{c}$ provide n is a positive integer. Of course, if $n = 0$ this cardinality is 1. It then follows that the set of all finite subsets of $\mathbf{R}$ has cardinality equal to $\aleph_0 \cdot \mathbf{c} + 1 = \mathbf{c}$.

To complete the proof it will suffice to show that the set $\mathbf{D}$ of all countably infinite subsets of $\mathbf{R}$ also has cardinality $\mathbf{c}$. We can define a 1–1 map from $\mathbf{R}$ into this set as before, sending r_0 to the set of all numbers of the form $r_0 + k$ where k is a nonnegative integer. This mapping is injective because each set in the range has a least element and for different real numbers one obtains different least elements.

Thus it only remains to show the cardinality of this set of subsets is less than or equal to $\mathbf{c}$.

Suppose that B is a countably infinite subset of $\mathbf{R}$. Then there is a 1–1 correspondence from $\mathbf{N}$ to B, so we pick such a mapping $h_B : \mathbf{N} \to B$ (*we are using the Axiom of Choice to do this*). We may now compose this chosen bijection with inclusion to obtain a mapping f_B from $\mathbf{N}$ into mapping from $\mathbf{D}$ to $\mathbf{F}(\mathbf{N}, \mathbf{R})$.

If we take different subsets we obtain different mappings because their ranges are unequal, and this means that there is a 1–1 map from $\mathbf{D}$ to $\mathbf{F}(\mathbf{N}, \mathbf{R})$, so that $|\mathbf{D}| \leq |\mathbf{R}|^{|\mathbf{N}|}$. Since we already know that $\mathbf{c} \leq |\mathbf{D}|$, everything reduces to proving that $|\mathbf{R}|^{|\mathbf{N}|} = \mathbf{c}$. This is a consequence of the following chain of equations:

$$\left(2^{\aleph_0}\right)^{\aleph_0} \quad = \quad 2^{\aleph_0 \cdot \aleph_0} \quad = \quad 2^{\aleph_0}$$

It follows that the set of countably infinite subsets of $\mathbf{R}$ has the same cardinality as $\mathbf{R}$ itself.■

5. This is very similar to the preceding example. Let X denote the set of continuous real valued functions on the unit interval, and let Y denote the set of functions defined at rational points of that interval. There is a natural map from X to Y defined by restricting to the rational points of the interval, and the statement in the exercise means that this mapping is injective. The reasoning of the previous problem shows that $|Y| = \mathbf{c}$ and hence that $|X| \leq \mathbf{c}$. On the other hand it is easy to show that $\mathbf{c} \leq |X|$; for example, we may define a 1–1 mapping from $\mathbf{R}$ into X sending r to the constant function whose value at every point is equal to r. Therefore it follows that $|X| = \mathbf{c}$.■

VI.6 : Transfinite induction and recursion

Exercises to work

1. It will be necessary to assume Axiom of Choice and Well-Ordering Principle from Section VII.1 of the lecture notes.

Let A be the original partially ordered set, and let P be a well-ordered set which is in 1–1 correspondence with $\mathbf{P}(A)$. Let $A^+ = A \cup \{A\}$ and extend the partial ordering on A to A^+ by making $A \in A^+$ the maximal element. We shall define a nondecreasing map $f : P \to A^+$ by transfinite recursion such that f is strictly increasing on $f^{-1}[A]$.

Denote the minimal element of P by 0, and define $f(0)$ by picking a point in A using a choice function. Suppose now that we have defined $f(\beta)$ for all $\beta < \alpha$; we need to define $f(\alpha)$. There are two cases. If there is some $z \in A$ such that $z > f(\beta)$ for all $\beta < \alpha$, define $f(\alpha)$ by choosing such a value of z (again, this requires a choice function). If no such value of z exists, let $f(\alpha) = A$.

Let $B = f[\,f^{-1}[A]\,]$; since P is well-ordered and f is strictly increasing on on $f^{-1}[A]$, it follows that B is a well-ordered subset of A. Thus it will suffice to show that B is cofinal in A. Suppose that $x \in A$; we need to show that there is some $b \in B$ such that $b > x$. Assume this does not hold for some particular choice of x. If this happens then the recursive definition yields a strictly increasing map from P into A, and in fact the image is contained in the set of all elements less than x. Since f is strictly increasing it follows that $|P| \leq |A|$. However, by construction we have $|P| = |\mathbf{P}(A)| > |A|$, which yields a contradiction. This means that for each $x \in A$ there must be some b such that $b > x$, so that B is a cofinal well-ordered subset.∎

2. Let A be the linearly ordered subset. If A is well-ordered, then the conclusion of the exercise is true because every nonempty subset of a well-ordered set is well-ordered. — Conversely, suppose that for each $x \in A$ the set of all strict predecessors of A is well-ordered, and let B be a nonempty subset of A. We need to show that B has a minimal element. Let $b_0 \in B$; if b_0 is a minimal element of B, we are done. On the other hand, if b_0 is not a minimal element and $L(b_0)$ is the set of strict predecessors of b_0, then $B \cap L(b_0)$ is nonempty, and since $L(b_0)$ is well-ordered it follows that $B \cap L(b_0)$ has a minimal element b_1. We claim that b_1 is a minimal element of B. For each $y \in B$ we have $y = b_0$, $y > b_0$ or $y < b_0$. In the first two cases we have $y \geq b_0 > b_1$, and in the last case we have $b_1 \leq y$ because then $y \in B \cap L(b_0)$ and b_1 is a minimal element of the intersection.∎

3. Let A be a well-ordered set, and let $A^{\mathbf{op}}$ denote A with the reverse ordering. If A is infinite, then A contains a subset that has the same order type as

$$\omega \;:=\; \{\,0 \;<\; 1 \;<\; 2 \;<\; 3 \;<\; 4 \;<\; 5 \;<\; 6 \;\cdots\}$$

and since $\omega^{\mathbf{op}}$ does not have a minimal element it follows that $A^{\mathbf{op}}$ is not well-ordered. — Suppose now that A is finite. In order to prove that $A^{\mathbf{op}}$ is well-ordered, we need to show that every nonempty subset of A has a maximal element.

We shall prove this by induction on $|A|$. If $|A| = 0$ the statement is vacuously true. Similarly, if $|A| = 1$, then A has a unique nonempty subset, and its unique element is a maximal element. Suppose now that we know the result for $|A| = n \geq 1$, and suppose that B is a well-ordered set with $(n+1)$ elements. Let 0 be the minimal element of B, and let $B_1 = B - \{0\}$. Given a nonempty subset $C \subset B$, let $C_1 = C \cap B_1$. If C_1 is nonempty, then by the induction hypothesis it follows that C_1 has a maximal element m. Since $C \subset C_1 \cup \{0\}$ and $0 < m$, it follows that m is also a maximal element of C. On the other hand, if $C_1 = \emptyset$ then C must be equal to $\{0\}$ and hence 0 is the maximal element of C.■

VII. The Axiom of Choice and related topics

VII.1 : Nonconstructive existence statements

Exercises to work

1. Define $g : B \to A$ on $f[A]$ by taking $b \in f[A]$ and picking $g(b) \in A$ such that $f(g(b)) = b$, and define g on $B - f[A]$ by setting $g(b) = z$ for some chosen element $z \in A$. We need to show that $f = f \circ g \circ f$. By construction, if $a \in A$, then $g(f(a))$ satisfies $f(g(f(a))) = f(a)$, so the condition $f = fgf$ is satisfied.■

2. If $|A| \leq |B|$ then there is a 1–1 mapping $f : A \to B$, and by Exercise III.4.13 there is a mapping $g : B \to A$ such that $g \circ f = \mathrm{id}_A$. The mapping g is onto because $a \in A$ can be written as $g(b)$ where $b = f(a)$. — Conversely, if there is a surjection $f : B \to A$, then by the Axiom of Choice there is a function $s : A \to B$ such that $s(a) \in f^{-1}[\{a\}]$ for all $a \in A$. It follows that $f \circ s(a) = a$ for all a. We claim that s is 1–1; if $s(u) = s(v)$, then $u = f(s(u)) = f(s(v)) = v$. Therefore we have $|A| \leq |B|$.■

3. To see that $W_r \cap W_q = \emptyset$ if $q \neq r$, observe that the second coordinates of elements in W_r are all equal to r while the second coordinates of elements in W_r are all equal to q. Therefore the second coordinates of elements of W_r and W_q are distinct, so that $W_r \cap W_q = \emptyset$. — The union of the sets $W_q = \cup_k Y_k \times \{k\}$ is equal to $\amalg_k Y_k$ by the definition of the latter. — For each q there is a 1–1 correspondence between Y_q and $W_q = Y_q \times \{q\}$ sending y to (y, q).■

4. For each k we are given a bijection $f_k : Y_k \to V_k$; denote the respective inverses by g_k. If we define $f : \amalg_k Y_k \to \amalg_k V_k$ by $f(y, k) = (f_k(y), k)$, then the map $g : \amalg_k V_k \to \amalg_k Y_k$ defined by $g(v, k) = (g_k(v), k)$ satisfies $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$, so that g is an inverse to f and both maps are bijections.■

VII.2 : Extending partial orderings

Exercises to work

1. The standard alphabetical ordering is a linear ordering that contains the given partial ordering. One way to visualize this is to move the pieces of the Hasse diagram slightly so that a falls below b, etc. — One can check this more methodically by constructing a matrix whose rows and columns correspond to the points of the original set in alphabetical order. Saying that the usual alphabetical ordering contains the given one amounts to saying that all ordered pairs for which the original relation holds must lie on or above the main diagonal. The following chart indicates this. In the latter some elements on or above the main diagonal are marked with numbers. If a number appears in the position (x, y), it means that xRy and there is a chain of length k such that $xRx_1R \cdots Rx_k = y$; if nothing appears, then no such chain exists.

	a	b	c	d	e	f	g	h	i	j	k	l	m
a	0			1				2	2	3	3	4	4
b		0		1	1			2	2	3	3	4	4
c			0			1	2				3	4	4
d				0				1	1	2	2	3	3
e					0			1		2	2	3	3
f						0	1				2	3	3
g							0				1	2	2
h								0		1	1	2	2
i									0	1		2	
j										0		1	
k											0	1	1
l												0	
m													0

2. The usual ordering works because if a and b are positive integers such that a divides b, then $a \leq b$.■

3. One way of finding a suitable linear ordering is to draw a Hasse diagram as in Exercise 1. The file `hasse-VII-2-3.JPG` in the online directory depicts one such possibility; namely, the linear ordering given by the following chain:

$$A < G < B < L < C < H < D < M < K < E < F$$

As before, one way to visualize this is to move the pieces of the Hasse diagram slightly. More methodically, this can be checked by constructing a matrix whose rows and columns correspond to the points of the original set in the order listed above. Saying that the new linear ordering contains the given partial ordering amounts to saying that all ordered pairs for which the original relation holds must lie on or above the main diagonal.

Here is the chart which corresponds to the one in Exercise 1; the notation for the entries of the chart is the same as for the earlier exercise.

	A	G	B	L	C	H	D	M	K	E	F
A	0	1	1	1	2	2	3	2	3	4	3
G		0			1	1	2		3	3	4
B			0		1		2			3	4
L				0				1			2
C					0		1			2	3
H						0	1		1	2	2
D							0			1	2
M								0			1
K									0		1
E										0	1
F											0

4. The main thing to do is to order the subsets so that the subsets with p elements come before the subsets with q elements if $p < q$. For example, this can be done as follows:

$$\emptyset \ < \ \{1\} \ < \ \{2\} \ < \ \{3\} \ < \ \{1,2\} \ < \ \{1,3\} \ < \ \{2,3\} \ < \ \{1,2,3\}$$

One interesting exercise might be to determine exactly how many linear orderings contain the given partial ordering.∎

5. This is similar to Exercise 1. Once again, the standard alphabetical ordering is a linear ordering that contains the given partial ordering, and one way to visualize this is to move the pieces of the Hasse diagram slightly so that a falls below b, etc. — Once again, it is possible to check this more methodically by constructing a matrix whose rows and columns as in Exercise 1. Here is what one obtains for the exercise we are now considering:

	a	b	c	d	e	f	g	h	i	j	k	l
a	0				1			2		3		4
b		0			1	1		2	2	3	3	4
c			0			1	1		2	3	3	4
d				0			1		2	3	3	4
e					0			1		2		3
f						0			1	2	2	3
g							0		1	2	2	3
h								0		1		2
i									0	1	1	2
j										0		1
k											0	1
l												0

VII.3 : Equivalence proofs

Exercises to work

1. Let **F** be a (nonempty) family of subsets of A of finite character. We want to apply Zorn's Lemma to this family.

In order to do so, we need to show that linearly ordered subsets of **F** have upper bounds in **F**. Suppose that $\mathbf{L} \subset F$ is a linearly ordered subfamily consisting of the sets L_α. We claim that $M = \cup_\alpha L_\alpha$ also belongs to **F**, and we shall prove this using the finite character assumption.

Suppose that $C \subset M$ is finite with elements $c_1 \ , \cdots \ , c_k$. Then we can find $L_j \in \mathbf{L}$ such that $c_j \in L_j$ for all j. Given any finite subset of a linearly ordered set, it is always possible to find a maximal element; applying this to the present situation, we can find some m such that L_m contains L_j for all j. It follows that $C \subset L_j$, so the finite character assumption implies that $C \in \mathbf{F}$. Thus we have shown that every finite subset of M belongs to **F**, and since the latter has finite character it follows that M itself belongs to **F**. As noted before, one can now apply Zorn's Lemma to find a maximal subset in **F**.∎

2. The hypotheses should have included an assumption that all the subsets in $\mathbf{F}$ are nonempty (the conclusion makes no sense if A is the empty set). — Let $P_+(S)$ be the set of all nonempty subsets of S, and let $c : P_+(S) \to S$ be a choice function on $P_+(S)$. Let $C \subset S$ be the image of $c|\mathbf{F}$. Suppose now that $A \subset \mathbf{F}$. Then we know that $c(a) \in C \cap A$. On the other hand, if $x \in C \cap A$, then $x = c(b)$ for some subset B, and it follows that $x \in B$ as well. Since $A \cap B = \emptyset$ if $A \neq B$ (this is the condition on $\mathbf{F}$), it follows that x must be equal to $c(a)$, and therefore we know that $C \cap A = \{ c(a) \}$.■

VII.4 : Additional consequences

Exercises to work

1. Once again we shall follow the hint, starting by choosing X_i and Y_j such that $|X_i| = \alpha_i$ and $|Y_j| = \beta_j$. We need to prove that there is no surjection from $\coprod_i X_i$ to $\prod_i Y_i$. In other words, given an injective mapping $f : \coprod_i X_i \to \prod_i Y_i$, we need to find a point in the codomain which does not lie in the image of f.

For each i let h_i be the composite

$$X_i \longrightarrow \coprod_j X_i \longrightarrow \prod_j Y_j \longrightarrow Y_i$$

where the first map is the standard injection of X_i into the disjoint union and the last map is the projection onto Y_i. The cardinality inequality implies that h_i cannot be surjective, so there is some $y_i \in Y_i$ which does not lie in the image of h_i. Let $\mathbf{y} \in \prod_i Y_i$ be the element whose coordinates are given by the corresponding elements y_i.

However, if $\mathbf{y}$ did lie in this image, then for some k in the indexing set J the element $\mathbf{y}$ would be the image of an element coming from X_k. This would mean that the coordinate y_k would be equal to $h_k(x)$, where $x \in X_k$ and the image of x in $\amalg_j X_j$ maps to $\mathbf{y}$. Since y_k does not lie in the image of h_k by construction, it follows that $\mathbf{y}$ cannot lie in the image of f. Therefore we know that

$$\sum_i \alpha_i \not\geq \prod_i \beta_i$$

and by the linear ordering property for cardinal numbers it follows that the number on the left hand side is strictly less than the number on the right hand side.■

2. Under the conditions of this exercise one can prove the following weaker inequality:

$$\sum_i \alpha_i \leq \prod_i \beta_i$$

PROOF. As suggested by the hint, the first step is to note that if α is an infinite cardinal number, then we have

$$\alpha \leq \alpha + 1 \leq \alpha + \aleph_0 = \alpha$$

because $\alpha + \aleph_0 = \alpha$ for every infinite cardinal number, so that $\alpha + 1 = \alpha$ also holds.

As in the preceding exercise, choose sets X_i and Y_j such that $|X_i| = \alpha_i$ and $|Y_j| = \beta_j$. As elsewhere, we let $\sigma(C) = C \cup \{C\}$ and use the fact that $C \notin C$ to conclude that $\sigma(C)$ is given by C

plus some point which does not lie in C. Since all cardinal numbers in sight are infinite, it follows that for all i the sets $\sigma(X_i)$ amd $\sigma(Y_i)$ have the same cardinalities as X_i and Y_i respectively. The assumption that $\alpha_i \leq \beta_i$ for all i implies that we have injections $g_i : X_i \to Y_i$.

We shall now define a mapping

$$F : \prod_j \sigma(X_j) \longrightarrow \prod_j \sigma(Y_j)$$

such that on the image of each X_i the map takes the point corresponding to $x \in X_i$ to the point in $\prod_j Y_j$ whose j^{th} coordinate equals $g_i(x)$ if $j = i$ and X_j if $j \neq i$. Since the images of X_u and X_v are disjoint if $u \neq v$, it follows that we obtain a well defined function in this manner.

It will suffice to prove that F is injective. Suppose that $F(p) = F(q)$. By definition, for all but one choice of indexing variable, the k^{th} coordinate of $F(p)$ is equal to the extra point $X_k \in \sigma(X_k)$, and a similar statement holds for $F(q)$. Therefore the exceptional coordinate is the same for both p and q. However, if ℓ is this exceptional coordinate, then by construction p and q both lie in the image of X_ℓ. The latter implies that $F(p) = F(q)$ if and only if $g_\ell(p) = g_\ell(q)$. Since g_ℓ is injective, it follows that $p = q$. Therefore F is 1–1 as required.■

3. 9*i*) Let A, B, C be sets such that $|A| = \alpha$, $|B| = \beta$, and $|C| = \gamma$. The condition $\alpha \leq \beta$ means there is an injection $j : A \to B$. Define an associated map of function sets

$$j_\# : \mathbf{F}(C, A) \longrightarrow \mathbf{F}(C, B)$$

by the formula $j_\#(f) = j \circ f$. The assertion about cardinal numbers will follow if we can prove that $j_\#$ is injective.

Suppose that $f_1, f_2 \in \mathbf{F}(C, A)$ satisfy $j_\#(f_1) = j_\#(f_2)$. Then $j \circ f_1 = j \circ f_2$, so that $j(f_1(x)) = j(f_2(x))$ for all $x \in C$. Since j is injective, this means that $f_1(x) = f_2(x)$ for all $x \in C$, which in turn implies that $f_1 = f_2$. Therefore the map $j_\#$ is injective as required.■

(*ii*) One way to do this is to start with the special cases where α and β are (finite) powers of 2. More precisely, if $\beta = 2^k$ for some $k \geq 1$ we shall prove that $\beta^\gamma = 2^\gamma$.

By the transfinite laws of exponents the left hand side is equal to $2^{k\gamma}$, and since $k \cdot \gamma = \gamma$ for every positive integer k and transfinite cardinal γ, the desired conclusion follows immediately when β is a positive integral power of 2.

For general choices of $\beta \geq 2$ we can find powers of 2 such that

$$2^p = \beta_0 \leq \beta \leq \beta_1 = 2^q$$

and if we combine this with the first part of the exercise, we see that

$$2^\gamma \leq \beta_0^\gamma \leq \beta^\gamma \leq \beta_1^\gamma = 2^\gamma$$

where the first and last equations follow from the discussion in the preceding paragraph. We can now use the Schröder-Bernstein Theorem to conclude that $\beta^\gamma = 2^\gamma$.■

4. Suppose that we have an infinite non-limit ordinal $\mu + 1$ and a 1–1 correspondence between the corresponding set $S[\mu + 1]$ and some other set X. Then we have the cardinal number identities

$$|X| = \big|S[\mu + 1]\big| = \big|S[\mu]\big| + 1 = \big|S[\mu]\big|$$

so that there is a 1–1 correspondence between X and the elements of the ordinal number μ. Therefore $\mu + 1$ cannot be the least ordinal for which there is a 1–1 correspondence, and it follows that if λ_0 is the minimum such ordinal, then λ_0 must be a limit ordinal.■

5. All finite ordinals are cardinal numbers, and the first infinite ordinal ω is equal to $\aleph_0$. Thus the next infinite ordinal, namely $\omega + 1$, is the first ordinal number that is not also a cardinal number.■

6. We shall solve this in a sequence of steps:

First, we shall use a previous exercise to prove the result when $|A| \leq 2^{\aleph_0}$.

Let $D(A)$ denote the countably infinite subsets. Since A contains a countably infinite subset B, it follows that $2^{\aleph_0} = |D(B)| \leq |D(A)|$, and since there is an injective mapping from A to $\mathbf{R}$ it also follows that $|D(A)| \leq D(\mathbf{R})| = 2^{\aleph_0}$, Therefore the number of countably infinite subsets is $2^{\aleph_0}$ by the Schröder-Bernstein Theorem.

From this point on assume that $|A| \geq 2^{\aleph_0}$. *Next, we shall prove that the set of countably infinite subsets is in* $1-1$ *correspondence with the set* $A^{\mathbf{N}}$ *of functions from* $\mathbf{N}$ *to* A *as follows: Given a countably infinite subset* E *and a specific* $1-1$ *correspondence from* $\mathbf{N}$ *onto* E *we shall obtain a map from* $\mathbf{N}$ *to* A *that turns out to be* $1-1$.

Since the image of the map associated to a subset B is equal to B by construction, it follows that different subsets determine functions with different images. Thus the functions must also be different.

By the Schröder-Bernstein Theorem it will be enough to define a map from $A^{\mathbf{N}}$ *to countably infinite subsets of* A. *There is a* $1-1$ *map from such functions to countable subsets of* $\mathbf{N} \times A$ *given by taking the graphs of functions. We shall use this to define the desired map to countable subsets of* A*? A comparison of* $|A|$ *and* $|\mathbf{N} \times A|$ *will be helpful here.*

Since A is infinite, the rules for transfinite cardinal arithmetic imply that $|A| = |\mathbf{N} \times A|$. Thus it is also enough to prove that the number of countably infinite subsets of $\mathbf{N} \times A$ is at least as large as the cardinality of $A^{\mathbf{N}}$. But given two functions from $\mathbf{N}$ to A, their images are distinct countably infinite subsets of the product $\mathbf{N} \times A$. Note that the graphs are always infinite because for each $n \in \mathbf{N}$ we have a point in $\mathbf{N} \times A$ whose first coordinate is equal to n.

Finally, we shall use a modified version of the Zorn's Lemma argument proving $\alpha \cdot \alpha = \alpha$ *for infinite cardinals* α *to prove that* $\alpha^\omega = \alpha$. *Specifically, we shall consider the collection of all pairs* (B, φ) *consisting of* $B \subset A$ *satisfying* $|B| \geq 2^{\aleph_0}$ *and a bijection* $\varphi : B^{\mathbf{N}} \to B$, *with a partial ordering such that* $(B, \varphi) \leq (D, \psi)$ *if and only if* $B \subset D$ *and* $\psi = \varphi$ *on* $B^{\mathbf{N}}$. *We may use the previously established fact that* $2^{\mathbf{c}} = \mathbf{c}^{\mathbf{c}}$ *(where* $\mathbf{c} = 2^{\aleph_0}$*) from the proof of Exercise* **rm VI.4.3** *to show this set is nonempty.*

By assumption A has a subset B such that $|B| = |\mathbf{R}|$, and we know there is a 1–1 correspondence between $\mathbf{R}$ and $\mathbf{R}^{\mathbf{N}}$ by the earlier exercise.

We shall verify that Zorn's Lemma applies and hence there is a maximal pair, say (B, φ).

Given a linearly ordered collection $(B_\alpha, \varphi_\alpha)$, we need to show that their union belongs to the given collection. Let $B^* = \cup B_\alpha$; then it follows immediately that one obtains a well defined mapping $\varphi^* : (B^*)^{\mathbf{N}} \to B^*$ from the mappings φ_α, so one needs to check that this map is bijective. To see it is injective, suppose $x, y \in (B^*)^{\mathbf{N}}$. Then there is some α such that $x, y \in (B_\alpha)^{\mathbf{N}}$, and $\varphi^*(x) = \varphi^*(y)$ implies $\varphi_\alpha(x) = \varphi_\alpha(y)$. Since φ_α is injective, this means $x - y$. To see that φ^* is surjective, let $z \in B^*$, so that $z \in B_\alpha$ for some α and hence lies in the image of φ_α, which means it also lies in the image of φ^*. Thus we have shown that in our partially ordered set, every linearly ordered subset has an upper bound, and this means that Zorn's Lemma applies.

If $|B| = |A|$ we are done, so suppose instead that $|B| < |A|$. In this case we shall show there is some $C \subset A$ such that $C \subset A - B$ and $|C| = |B|$.

If this were false then the cardinality of every subset of $A - B$ would be strictly less than $|B|$, and in particular $|A - B| < |A|$, so that $|A|$ would be less than $|B|$, which is less than $|A|$. This contradiction shows that there must be some subset of $A - B$ whose cardinality is equal to $|B|$. In fact, one has $|A - B| = |A| > |B|$ in our situation, but we shall not need the full strength of this conclusion.

We now explain why $(B \cup C)^{\mathbf{N}}$ can be written as a union of pairwise disjoint subsets S_Y, where Y runs over all subsets of $\mathbf{N}$ such that $\mathbf{a} \in S_Y$ if and only if $a_k \in B$ for $k \in Y$ and $a_k \in C$ otherwise. We shall show that each such set in 1–1 correspondence with $B^{\mathbf{N}}$ and $C^{\mathbf{N}}$.

When we write the product as a union of the pairwise disjoint subsets S_Y, we are merely sorting the elements of the product into subsets depending upon which coordinates lie in B and which lie in C. Since B and C are disjoint, these two properties are mutually exclusive. Each of the sets in question is a product for which every factor is either B or C. Therefore all the factors are in 1–1 correspondence with both B and C, and it follows (from an exercise in Section V.1) that every set S_Y is in 1–1 correspondence with $B^{\mathbf{N}}$ and $C^{\mathbf{N}}$

If $\mathbf{P}_1(\mathbf{N})$ denotes the proper subsets of $\mathbf{N}$, we shall construct a bijection from $\mathbf{P}_1(\mathbf{N}) \times C$ to C.

The set $\mathbf{P}_1(\mathbf{N})$ is obtained from the entire power set $\mathbf{P}(\mathbf{N})$ by deleting one subset, and since we are working with infinite sets the cardinalities of $\mathbf{P}_1(\mathbf{N})$ and $\mathbf{P}(\mathbf{N})$ are equal. Since $|B| = |C| \geq |\mathbf{P}(\mathbf{N})|$, it follows that $|\mathbf{P}_1(\mathbf{N}) \times C| = |\mathbf{P}(\mathbf{N}) \times C| = |C|$.

We shall show there is a bijection from $(B \cup C)^{\mathbf{N}}$ to $B \cup C$ sending $S_{\mathbf{N}} = B^{\mathbf{N}}$ to B by the maximal map and sending the other sets S_Y to C by the composites of $S_Y \to \{Y\} \times C$ and $\mathbf{P}_1(\mathbf{N}) \times C \to C$.

It is only necessary to define the bijection on the pieces. The assertion gives the definition on $S_{\mathbf{N}} = B^{\mathbf{N}}$, and it describes the map on the remaining pieces as well. We need to check this map is bijective. It will suffice to show that the partial composite $\cup_{Y \neq \mathbf{N}} S_Y \to \mathbf{P}_1(\mathbf{N}) \times C$ is bijective because the total composite sends the codomain to C, which is disjoint from B.

By construction the map sends the pairwise disjoint subsets S_Y into the pairwise disjoint subsets $\{Y\} \times C$, so the proof of the bijectivity assertion reduces to verifying the latter for each of the pieces we have described. But on these pieces the map is a bijection by construction.

We claim this a contradiction, and we shall determine the source of the contradiction.

We have constructed a bijection from $(B \cup C)^{\mathbf{N}}$ to $B \cup C$ which properly contains a maximal bijection. The problem arose in our assumption that $|B|$ was strictly less than $|A|$, so the latter must be false and we must have $|B| = |A|$. As noted before, this proves the identity $|A^{\mathbf{N}}| = |A|$ when $|A| \geq |\mathbf{R}|$ and thus also completes the proof of the exercise.∎

7. Since every set can be well-ordered, this is true for $\mathbf{R}$ in particular. Therefore the set of all uncountable ordinals is nonempty, so it must contain a least element, which we are calling Λ_1 here.

To prove the assertion about least upper bounds, we use the standard von Neumann model for the ordinals (ordered by $\alpha < \beta \Longleftrightarrow \alpha \in \beta$). Suppose that we have a countable family of ordinals $\alpha_k \in \Lambda_1$ and we consider the union C of all ordinals β such that $\beta \leq \alpha_k$ for some k. Now each α_k is countable, si this means that C is countable. By construction, if $\gamma \leq \beta$ for some $\beta \in C$, then $\gamma \in C$. Now $\Lambda_1 - C$ cannot be countable since C is countable but Λ_1 is not. Therefore $\Lambda_1 - C$ is nonempty and as such has a least element. By a previous sentence we know that this least element δ_0 must be strictly greater than every element of C and hence δ_0 is an upper bound for the set of all ordinals α_k.

To conclude we must find a least upper bound for this set of ordinals. Suppose that δ_0 is not a least upper bound. Then there is some $\delta_1 < \delta_0$ that is also an upper bound. We claim that δ_1 must be a least upper bound. By construction of δ_0 we know that anything strictly less than δ_0 is not greater than every element of C. Hence there is some $\gamma \in C$ such that $\delta_1 \leq \gamma < \delta_0$. Now $\gamma < \alpha_m$ for some m and by the defining properties of δ_0 we also have $\delta_1 \leq \alpha_m < \delta_0$. On the other hand, since δ_1 is an upper bound we also have the reverse inequality $\alpha_m \leq \delta_1$ so that equality must hold. Thus we have shown that $\alpha_m \geq \alpha_k$ for all k, which means that α_m must be the least upper bound for the original set of ordinals.■

Postscript. In fact, if δ_0 is not the least upper bound, then we have $\delta_0 = \alpha_m + 1$ because δ_0 is the least element that is greater than each of the elements in C.■

8. We shall need the following elementary fact about linearly ordered sets:

LEMMA. *If Y is a linearly ordered set and $y_1, \cdots, y_n \in Y$, then there is some k such that $y_k \geq y_j$ for all j.*

Proof. This is trivial if $n = 1$; assume it is true for $n = m$, suppose we are given $y_1, \cdots, y_{m+1} \in Y$, and let $Y_0 = Y - \{y_{m+1}\}$. Then by the induction hypothesis there is some q such that $y_q \geq y_j$ for $j \leq m$. Since Y is linearly ordered we know that either $y_q \leq y_{m+1}$ or $y_{m+1} = y_q$. In the first case it follows that $y_{m+1} \geq y_j$ for all $j \leq m+1$, and in the second it follows that $y_q \geq y_j$ for all $j \leq m+1$.■

Solution to the exercise. Let F be a family of subsets of some set S with the finite intersection property, and let $\mathcal{G}$ be the set of all families G of subsets of S such that $F \subset G$ and G has the finite intersection property; then $\mathcal{G}$ is partially ordered with respect to inclusion. The proof of the statement in the exercise reduces to showing that the hypothesis in Zorn's Lemma is true for $\mathcal{G}$.

Let $\mathcal{L}$ be a nonempty linearly ordered subset of $\mathcal{G}$, and let $L^* = \cup\{L \in \mathcal{L}\}$. Clearly $L \subset L^*$ for all $L \in \mathcal{L}$; we claim that $L^* \in \mathcal{G}$; *i.e.*, $F \subset L^*$ and L^* has the finite intersection property. The first statement is clear since F is contained in every $L \in \mathcal{L}$. To prove the second, suppose we are given $A_1, \cdots, A_n \in \mathcal{L}^*$. For each j there is some $L_j \in \mathcal{L}$ such that $A_j \in L_j$. Since $\mathcal{L}$ is linearly ordered, the lemma shows there is some q such that $L_j \subset L_q$ for all j. Therefore we have $A_1, \cdots, A_n \in L_q$, and since L_q has the finite intersection property we conclude that $\cap_j A_j \neq \emptyset$. — We have now shown that $\mathcal{G}$ is a partially ordered set in which linearly ordered subsets have upper bounds, and therefore $\mathcal{G}$ has a maximal element G^* by Zorn's Lemma. By construction G^* is a maximal family of subsets with the finite intersection property such that $F \subset G^*$.■

Further comment. In many uses of Zorn's Lemma, it is important to understand what maximality implies for a set H which properly contains the maximal set G^*. In this problem, it means that one can find a finite collection of subsets B_t in H such that $\cap_t B_t = \emptyset$. Here are two other facts about the maximal family G^* in the exercise that are true: (i) The family G^* is closed under finite intersections. (ii) If we are given $A \in G^*$ and $C \subset S$ such that $A \subset C$, then $C \in G^*$. — Both of these follow because $G^* \cup \{C\}$ has the finite intersection property; writing up this argument in detail is left to the reader.■

Hasse diagram for Exercise VII.2.3