Math 145A
Winter 2020

UPDATED GENERAL INFORMATION — JANUARY 30, 2020

The first midterm examination

The first midterm examination, which will take place on **Wednesday, February 5**, will cover Chapters 4 and 5 in Sutherland and the corresponding material in the course directory.

The main points to understand are the definitions of fundamental concepts including least upper bounds, metric spaces, continuity, and open sets. These should be understood well enough that they can be applied to specific classes of examples, some concrete and others (not too) abstract. For example, you might be asked to verify that a specific type of function is continuous or a specific type of subset in a metric space is either open or not open.

The problems on the exam will be similar to the easy and moderately challenging exercises. Here are a few sample questions to consider. Some are probably more demanding than the problems which will appear on the exam but not dramatically so. Solutions will be posted by tomorrow afternoon.

When working the first two problems, the following characterization for the least upper bound of a nonempty subset $A \subset \mathbb{R}$ will probably be useful: b is a/the least upper bound of A if (1) we have $a \leq b$ for all $a \in A$, (2) for every $\varepsilon > 0$ there is some $a \in A$ such that $a > b - \varepsilon$.

1. Let A be a nonempty subset of the reals for which M is a least upper bound, and let $c > 0$. If cA is the set of all real numbers expressible as ca for some $a \in A$, show that cA is bounded and find its least upper bound in terms of the quantities given in the problem.

2. Let A and B be nonempty bounded subsets of the reals with least upper bounds M and N respectively, and let C be the set of all numbers having the form $a + b$ where $a \in A$ and $b \in B$. Show that $M + N$ is the least upper bound for C.

3. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function which is not necessarily continuous. Define a new metric Δ on X by $\Delta(p, p') = d_X(p, p') + d_Y(f(p), f(p'))$. Verify that f defines a continuous function from (X, Δ) to (Y, d_Y).

4. Let $D \subset \mathbb{R}^2$ be the diagonal line consisting of all points having the form (x, x) for some $x \in \mathbb{R}$. Prove that D is not an open subset of $\mathbb{R}^2$ with respect to the usual metric(s) but its complement $\mathbb{R}^2 - D$ is open.

Suggested problems from 2019

Here are some problems that were suggested last year that are also worth considering. Problems marked with an asterisk involve material from Chapter 6 and can be skipped when reviewing for the midterm examination.

1. Let (X, d) be a metric space with only finitely many points. Prove that every subset of X is open.

2(*). Let (X, d) be a metric space, and let A be a subset of X which is both open and closed in X. Show that the complement $X - A$ is also both open and closed in X. Give an example of a metric space which contains a nonempty proper subset A which is both open and closed.

3(*). Let (X, d) be a metric space, and let $\{A_n\}$ be a sequence of subsets in X where n runs through all the positive integers. Give an example for which $\cup_n \overline{A_n} \neq \overline{\cup_n A_n}$, and determine whether containment in either direction is is always true. It is probably simplest to find an example in which $X = \mathbb{R}$ with the usual metric. Also, give an example of a sequence of distinct nonempty subsets in $\mathbb{R}$ for which equality does hold.

4(*). (a) Suppose that U is an open subset of $\mathbb{R}^n$ (with the Pythagorean metric). Show that every point of U is a limit point of U.

(b) Suppose that X is a finite metric space. Show that no point of X is a limit point of X.

(c) Find a countably infinite subset $D \subset \mathbb{R}^n$ (as above) such that every point of $\mathbb{R}^n$ is a limit point of D.

5. Suppose that we are given three metric spaces (X_i, d_i) where $i = 1, 2, 3$, and let D be the Pythagorean metric on the threefold Cartesian product $X_1 \times X_2 \times X_3$:

$$D\left((u_1, u_2, u_3),\ u_1, u_2, u_3) \right) \quad = \quad \left(\sum\ d_i(u_i, v_i)^2 \right)^{1/2}$$

Let q_j be the coordinate projection from $X_1 \times X_2 \times X_3$ to X_j where $j = 1, 2, 3$. If (Y, e) is a metric space and $f : Y \to X_1 \times X_2 \times X_3$ is a map of sets, prove that f is continuous if and only if each of the coordinate functions $f \circ q_j$ is continuous. [*Hint:* The threefold product with the Pythagorean metric is just the twofold product of $X_1 \times X_2$ with X_3 where the former has the Pythagorean product metric.]

6. In the setting of the preceding exercise, assume that all three metric spaces are the same object (X, d), and let $T : X^3 \to X^3$ be the map which cyclically permutes the coordinates: $T(u, v, w) = (v, w, u)$. Prove that T is continuous.

Suggested problems from 2017

And here are some problems that were suggested three years ago that are also worth considering.

1. Let $f : X \to Y$ be a function of sets, and let B be a subset of Y. Prove that

$$B \quad \subset \quad f^{-1}\left[\,[f[B]]\,\right]$$

and give an example for which the containment is proper.

2. Let $f(x) = 1/x$ on the interval $(0, 2)$, and let $\varepsilon > 0$. Fine $\delta > 0$ so that $|x - 1| < \delta$ implies $|f(x) - f(1)| < \varepsilon$. It might help to analyze this as follows: If $\varepsilon > 0$ and $\varepsilon < \frac{1}{2}$, for what values of $x \in (0, 2)$ do we have

$$1 \ - \ \varepsilon \quad < \quad \frac{1}{x} \quad < \quad 1 \ + \ \varepsilon\ ?$$

3. Let f be a monotonically increasing (but not necessarily strictly increasing) real valued function on the interval (a, b), and let $c \in (a, b)$. Define $f(c-)$ to be the least upper bound of all values $f(x)$ for $x < c$, and define $f(c+)$ to be the greatest lower bound of all values $f(x)$ for $x > c$. Prove that $f(c-) \leq f(c) \leq f(c+)$, and prove that f is continuous at c if and only if $f(c-) = f(c+)$.

4. In the real line give examples of subsets A, B satisfying the following conditions. [*Hint:* Try examples for which the subsets are intervals which may be open, closed or neither.]

(i) A is open, $A \cap B$ is open, but B is not open.

(ii) **(*)** A is closed, $A \cap B$ is closed, but B is not closed.

(iii) **(*)** Neither A nor B is closed, but $A \cap B$ is closed.

(iv) **(*)** Neither A nor B is closed, but $A \cup B$ is closed.

5(*). Let A be a subset of the real line, and assume b is a least upper bound for A. Prove that either $x \in A$ or else x is a limit point of A.

6(*). A subset A of a metric space X is said to be locally closed if it is the intersection of an open subset and a closed subset.

(i) If A is either open or closed in X, explain why A is locally closed in X.

(ii) Prove that $[0, 1)$ is a locally closed subset of the real line (and it is not locally closed by the reasoning on page 6.3 of the class notes). More generally, prove that every half-open interval $[a, b)$ or $(a, b]$ is a locally closed subset of the real line.

(iii) One can show that the only closed subset of the reals which contains the rationals is the entire real line. Using this, explain why the rationals are not a locally closed subset of the real line. [*Hint:* If $A = E \cap V$ where E is closed and V is open, why is $A \subset E$?]

7. Given two subsets A, B of a set X, the **symmetric difference** $A + B$ is the set of all points which are in either A or B but not in both. In symbols, it is defined by the equation

$$A \,+\, B \quad = \quad \Big(A \cap (X - B)\Big) \;\cup\; \Big(B \cap (X - A)\Big) \ .$$

If $f : Y \to X$ is a set-theoretic function, verify that $f^{-1}[A + B] \ \subset\ f^{-1}[A] + f^{-1}[B]$.

Note. The reason for the use of a plus sign is that this operation and intersection make the family of subsets of X into a commutative ring with unit.

8. (i) If $u, v \geq 0$ explain why $\sqrt{u + v} \ \leq\ \sqrt{u} \,+\, \sqrt{v}$. [*Hint:* Square both sides.]

(ii) If (X, d) is a metric space, show that $(X, \sqrt{d})$ is also a metric space.

(iii) More generally, if φ is a strictly increasing function from the nonnegative reals to themselves with $\varphi(0) = 0$ and $\varphi(u + v) \leq \varphi(u) + \varphi(v)$, show that $(X, \varphi \circ d)$ is also a metric space. [In particular, this holds if $\varphi(t) = Ct$ for some positive constant C.]

9. Suppose that X is a set and that d and d' are metrics on X. If $d^*(x_1, x_2)$ is the greater of $d(x_1, x_2)$ and $d'(x_1, x_2)$, prove that d^* defines a metric on X.

10. If X and Y are metric spaces, then a function $f : X \to Y$ is said to satisfy a *Lipschitz condition* if there is some constant $K \geq 0$ such that for all $x_1, x_2 \in X$ we have $d_Y(f(x_1), f(x_2)) \leq K \cdot d_X(x_1, x_2)$.

(i) If $f : [0, 1] \to \mathbb{R}$ is a mapping with a continuous derivative at all points of $[0, 1]$, show that f satisfies a Lipschitz condition on $[0, 1]$. [*Hint:* Use the Mean Value Theorem and the continuity of f'.]

(*ii*) Show that if $f : X \to Y$ is satisfies a Lipschitz condition then f is (uniformly) continuous.

Finally, it might be worthwhile to look at the older files `exam1w14key.pdf`, `exam1w16key.pdf`, `exam1w17key.pdf`, `exam2w17key.pdf`, and `exam1w19key.pdf`, which are copies of exams prepared for previous versions of this course; correct solutions to all problems are included in these files.