Math 145A
Winter 2020

UPDATED GENERAL INFORMATION — JANUARY 31, 2020

Solutions to study problems

The solutions to problems from previous years are in handwritten form. No solution to Problems 2 from 2017 is given because the details are already worked out in the file

`continuity-example.pdf`

and although no solution to Problem 7 is given in the handwritten pages there is a solution at the end of the file.

Solutions to the 2020 *problems*

1. We shall first solve this using the epsilon characterization in which accompanies the statement of the problem. First of all cM is an upper bound because $a \leq M$ (all $a \in A$) and $c > 0$ imply $ca \leq M$. Next, if $\varepsilon > 0$ then ε/c is positive, and therefore there is some $x \in A$ such that $x > M - (\varepsilon/c)$. Multiplying both sides of this by $c > 0$, we see that $cx > cM - \varepsilon$. Therefore cM satisfies the second property for a least upper bound as stated in the epsilon characterization.■

We can also verify this without using the epsilon characterization as follows. As before, we know that cM is an upper bound. Suppose now that N is an upper bound for cA. Then N/c is an upper bound for A and hence $M \leq N/c$, and if we multiply both sides by the positive constant c we find that $cM \leq N$.■

2. Every $c \in C$ has the form form $a + b$ where $a \in A$ and $b \in B$, and since A and B have least upper bounds M and N respectively, it follows that $c = a + b \leq M + N$. Therefore $M + N$ is an upper bound. To see that it is a least upper bound, let $\varepsilon > 0$. Then we also have $\frac{1}{2}\,\varepsilon > 0$, so by the epsilon characterization we can find $x \in A$ and $y \in B$ so that $x > M - \frac{1}{2}\,\varepsilon$ and $y > N - \frac{1}{2}\,\varepsilon$. It follows that $x + y \in C$ and

$$x + y \quad > \quad \left(M - \frac{\varepsilon}{2}\right) \; + \; \left(N - \frac{\varepsilon}{2}\right) \quad = \quad M + N - \varepsilon$$

and hence $M + N$ satisfies the epsilon condition for a least upper bound.■

3. Let $\varepsilon > 0$ and $p \in X$. We need to find some $\delta > 0$ such that $d_Y(f(p), f(p')) < \varepsilon$ when $\Delta(p, p') < \delta$. By definition the left hand side in the second inequality is $\Delta(p, p') = d_X(p, p') + d_Y(f(p), f(p'))$. Since both summands are nonnegative, it follows that $d_Y(f(p), f(p')) \leq \Delta(p, p')$. If the right hand side of this inequality is less than ε, it follows that the left hand side is also less than ε, and therefore we can take $\delta = \varepsilon$.■

4. To prove that D is not open it is enough to prove that for some $(x, x) \in D$ there is no $\delta > 0$ so that $N_\delta\left(\,(x, x)\,\right) \subset D$. In fact this is true for **every** such point, for $(x, x + \frac{1}{2}\,\delta) \in N_\delta\left(\,(x, x)\,\right)$ but $(x, x + \frac{1}{2}\,\delta) \notin D$.

We shall now verify that the complement $\mathbb{R}^2 - D$ is open. A point in the latter has the form (x,y) where $x \neq y$. If $\delta = \frac{1}{2}|x-y|$ and $(u,v) \in N_\delta\left((x,y)\right)$ it will suffice to show that $(u,v) \in \mathbb{R}^2 - D$. — It might be helpful to draw a picture in order to motivate this assertion; the main points to plot are (x,x), (x,y) and (y,y).

Assume to the contrary that there is a point $(w,w) \in N_\delta\left((x,y)\right)$. Since the distance between (x,y) and (u,v) is at least as large as both $|x-u|$ and $|y-v|$, if $u = v = w$ we see that both $|x-w|$ and $|y-w|$ are strictly less than δ. Therefore the Triangle Inequality implies that $|x-y|$ is strictly less than $2\delta = |x-y|$, which is a contradiction. The source of this contradiction is the assumption that there is a point $(w,w) \in N_\delta\left((x,y)\right)$, and hence no such point can exist. In other words, the neighborhood is contained in the complement of D.■

Comments on the handwritten pages

The so-called new problems are the ones from 2019, and the old problems are those from 2017.

Correction. In old problem 5, replace "$\{a\}$" with "$\{b\}$".

Comments on practice problems in

aabUpdate04.145A.w19.pdf

New problems

1. Let $X = \{x_1, \ldots, x_n\}$. If $x_i \in X$ let $a_i = \min_{j \neq i} d(x_j, x_i)$, so $a_i > 0$. Take $r = \frac{1}{2} a_i$. Then $N_r(x_i) = \{x_i\}$.

2. A open, closed in X.
   A open in $X \Rightarrow X - A$ closed.
   A closed in $X \Rightarrow X - A$ open.

3. Let $X = \mathbb{R}$, and take $A_n = \{\frac{1}{n}\}$, which is closed, so $\cup A_n = \cup \overline{A_n}$. However, $0 \in \overline{\cup A_n}$.
   Now $\overline{\cup A_n}$ is closed and contains A_k for all k, so $\overline{\cup A_n} \supseteq \overline{A_k}$ all k and hence $\overline{\cup A_n} \supseteq \cup \overline{A_n}$. One example for the last sentence is $A_n = \{n\}$, $n \in \mathbb{N}$.

4. (a) Let $x \in U$, let e be a unit vector, and choose $r > 0$ so $N_r(x) \subseteq U$. Then x is the limit of the sequence $x + \frac{r}{n} e$, and none of these $= x$.

(b) By #1 each $\{x\}$ is open in $X \Rightarrow$ $\{x\} = N_r(x)$ some $r > 0$. Hence $X \cap (N_r(x) - \{x\}) = X \cap \phi = \phi$ and by def x is not a limit point.

(c) Take $\mathbb{Q}^n \subseteq \mathbb{R}^n$.

5. Imitate the case of a twofold product. By construction $d_i(q_i(u), q_i(v)) \leq D(u,v)$ so $q_i : X_1 \times X_2 \times X_3 \longrightarrow X_i$ is continuous (given ε, take $\delta = \varepsilon$). Hence f cont. $\Rightarrow$ each $q_i f$ cont. Conversely, if each $q_i f = f_i$ cont and $\varepsilon > 0$ and $y_0 \in Y$ chose $\delta_i > 0$ so

$$d_Y(y_0, y) < \delta_i \Rightarrow d(f_i(y_0), f_i(y)) < \frac{\varepsilon}{\sqrt{3}}$$

Let $\delta = \min \delta_i$. Then $d(y, y_0) < \delta \Rightarrow$
$D(f(y), f(y_0)) < \varepsilon$.

6. Use #5 and note that
$$q_1 F = q_2, \quad q_2 F = q_3, \quad q_3 F = q_1.$$

Old problems

1. The containment follows because
$b \in B \Rightarrow f(b) \in f[B]$, so that $b \in f^{-1}[f[B]]$.
For the example with proper containment, take $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$, $B = [0, 1]$.
Then the right hand side is $[-1, 1]$.

2. PASS

3. By definition we have $f(b) < f(c_+) + \varepsilon$ for some $b > c$ and $f(a) > f(c_-) - \varepsilon$ for some $a < c$. Hence
$$f(c_-) - \varepsilon < f(x) < f(c_+) + \varepsilon$$
if $a < x < b$.

If $f(c-) = f(c+)$, then f is cont. at c;
take δ so $(c-\delta, c+\delta) \subseteq (a,b)$.
If $f(c_-) < f(c_+)$ there are sequences
$L_n \to c$ $(L_n \leq c)$ $\quad U_n \to c$ $(U_n \geq c)$
with $\lim f(L_n) = f(c-)$, $f(U_n) = f(c_+)$
and $f(c_-) \neq f(c+) \Rightarrow f$ not continuous at c.

4 (i)
$$\overset{A}{{}_0(\qquad\qquad)_1}$$
$$\underset{1/2}{(}\qquad B \qquad \underset{3/2}{]}$$

(ii)
$${}_0[\quad A \quad]_1$$
$$\underset{1/2}{[}\quad B \quad \underset{3/2}{)}$$

(iii)
$$\overset{0}{(}\quad A \quad \overset{1}{]}$$
$${}_{1/2}[\quad B \quad)^{3/2}$$

(iv)
$$\overset{0}{[}\quad A \quad \overset{2/3}{)}$$
$${}_{1/3}(\quad B \quad]_1$$

5. Suppose $b \in A$. Then if $\varepsilon > 0$ can find $a \in A$ so $b - a \leq \varepsilon$. Hence

$$(N_\varepsilon(b) - \{a\}) \cap A \neq \phi \text{ because}$$

$b \notin A$.

6. (i) A open $\Rightarrow A = X \cap A$ & X is closed
A closed $\Rightarrow A = X \cap A$ & X is open

(ii) $$[a, b) = [a, b+1] \cap (a-1, b)$$
$$(a, b] = (a, b+1) \cap [a-1, b].$$

(iii) Follow the hint. $Q = E \cap V \Rightarrow Q \subseteq E$, E closed. But then $E = \mathbb{R}$, so $Q = \mathbb{R} \cap V = V$, V open. Since V is not open this is impossible.

7. PASS

8. (i) $\sqrt{u+v} \leq \sqrt{u} + \sqrt{v}$ because

$(\sqrt{u+v})^2 = u + v$

$(\sqrt{u} + \sqrt{v})^2 = u + 2\sqrt{uv} + v$

and the first expression $\leq$ second.

(ii) The only nontrivial point to check is the Triangle Inequality. Do this verification using (i).

(iii) PASS

9. $d^* = \max d, d' \geq 0$ because $d, d' \geq 0$

If $d^* = 0$ then $\max d, d' = 0 \Rightarrow$
$d(u,v) = 0$ or $d'(u,v) = 0$. In either case
$u = v$.

$d^*(u,v) = d^*(v,u)$ because $d^* = \max d, d'$ and both of the latter have this property.

Finally

$d^*(x,z) = \max d(x,z), d'(x,z)$.

Say the max is $d(x,z)$.

Then $d^*(x,z) = d(x,z) \leq d(x,y) + d(x,z)$

$\leq d^*(x,y) + d^*(y,z)$ since $d \leq d^*$.

10. (i) Let $M = \max|f'|$ on $[0,1]$.

$0 < u < v < 1$. Then by the MVT

$|f(u) - f(v)| = |f'(W) \cdot (u-v)|$ where

$u < W < v$. The RHS $\leq M \cdot |u-v|$.

(ii) If $d(x_1, x_2) < \frac{\varepsilon}{K}$ then

$$d(f(x_1), f(x_2)) \leq K \cdot d(x_1, x_2) <$$

$$K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

Solution to old problem 7

Recall that the symmetric difference for two subsets C, D of a larger set S may be written in the form

$$C \; + \; D \quad = \quad \Big(C \cap (S - D)\Big) \,\cup\, \Big(D \cap (S - C)\Big)\,.$$

Now suppose that $f : Y \to X$ is a set-theoretic function and $A,\ B \subset X$. Since the inverse image construction preserves unions and intersections we have

$$f^{-1}[A + B] \quad = \quad f^{-1}\Big[A \cap (X - B)\Big] \,\cup\, f^{-1}\Big[(B \cap (X - A)\Big] \quad =$$

$$\big(\, f^{-1}[A] \cap f^{-1}[X - B]\,\big) \,\cup\, \big(\, f^{-1}[B] \cap f^{-1}[X - A]\,\big)$$

and since the inverse image construction also preserves relative complements the right hand side is equal to

$$\Big(f^{-1}[A] \,\cap\, (Y - f^{-1}[B])\Big) \,\cup\, \Big(f^{-1}[B] \,\cap\, (Y - f^{-1}[A])\Big)\,.$$

By definitiion the latter is equal to $f^{-1}[A] + f^{-1}[B]$.$\blacksquare$