UPDATED GENERAL INFORMATION — FEBRUARY 10, 2020

Recommended exercises for Chapter 6 *of Sutherland*

- Chapter 6: 6.3, 6.5, 6.6, 6.12, 6.13, 6.15, 6.20, 6.23

The following references are to the file file `exercises02w14.pdf` in the course directory.

- Additional exercises for Chapter 6: 1, 2, 4, 7

Reading assignments from solutions to exercises

Another recommendation is to read through the solution to Exercise 6.9 from Sutherland (see the file `solutions02w14.pdf` in the course directory). This exercise proves assertions in the notes about certain sets which arise in the study of double integrals (in multivariable calculus).

Intersections of empty families of sets

On page 7.4 of `math145Anotes07.pdf` there is a marginal note about considering an intersection of a family of subsets $\mathbf{F} = \{A_\alpha\}$ only when $\mathbf{F}$ is nonempty. This was also mentioned earlier in the course, and an undergraduate level discussion appears on pages 18–19 of Halmos, *Naive Set Theory*. The key point is that if the intersection of an empty family would be defined, then it would not be empty as one might expect, but instead it turns out to be everything (see also page 44 of the directory file `set-theory-notes.pdf`).

Assignments for Chapters $7-10$

Working the exercises listed below is **strongly recommended.**

The following exercises are taken from Sutherland:

- Chapter 7: 7.1 – 7.2, 7.6
- Chapter 8: 8.3, 8.4, 8.6
- Chapter 9: 9.11, 9.14
- Chapter 10: 10.1, 10.6, 10.10 – 10.11, 10.17 – 10.18 (in the latter, the map $f \times g$ should go from $X \times X'$ to $Y \times Y'$ and not from $X \times Y$ to $X' \times Y'$)

The following references are to the file `exercises03w14.pdf` in the course directory.

- Additional exercise for Chapter 7: 2

- Additional exercises for Chapter 8: 2 – 4

The following references are to the file `exercises04w14.pdf` in the course directory.

- Additional exercise for Chapter 9: 2
- Additional exercises for Chapter 10: 1 – 3

Reading assignments from solutions to exercises

Two additional strong recommendations are

(1) to read through the solutions to Additional Exercise 1 for Chapter 7 and Additional Exercise 5 for Chapter 8 in `exercises03w14.pdf`, both of which are worked out in the file `solutions03w14.pdf`.

(2) to read through the solutions to Exercises 9.5, 9.8, 10.12 and 10.16 from Sutherland, Additional Exercise 3 for Chapter 9 and Additional Exercise 4 for Chapter 10 in `exercises04w14.pdf`, all of which are worked out in the file `solutions04w14.pdf`.

Further reading recommendations

The files `intro2topA-08a.pdf` and `intro2topA-08b.pdf` explain the connection between the formal concept of homeomorphism and the often seen description of topology as a "rubber sheet geometry." Both contain color pictures which are meant to emphasize the geometric side of point set topology. In contrast, the files `intro2topA-07a.pdf` and `intro2topA-07b.pdf` concern the set-theoretic side of the subject. Thoroughout the subject, both sides play complementary roles: Most of the motivation and intuition is geometric, and the logical integrity of the subject is established using the techniques of set theory.

The second quiz

The second quiz on **Tuesday, February 19,** will cover the material corresponding to Chapters 6–8 of Sutherland. In addition to the starred problems in the file

`aabNewUpdate03.145B.w20.pdf`

Some practice problems (in addition to the previously recommended exercises) are given below:

New problems

1. Let X be a set, and let $\mathbf{T}(X)$ denote the set of all topologies on X. Explain why the unions and intersections

$$\bigcup_{\mathcal{U}\in\mathbf{T}(X)} \mathcal{U} \qquad \text{and} \qquad \bigcap_{\mathcal{U}\in\mathbf{T}(X)} \mathcal{U}$$

define topologies on X. [*Hint:* They are standard examples!]

2. (*a*) Let $(X,\mathbb{T})$ be a topological space such that for some fixed positive integer $n > 1$, the set X contains at least $n+1$ points and $\mathbb{T}$ contains every subset of X with exactly n points lies in $\mathbb{T}$. Prove that $\mathbb{T}$ is the discrete topology. [*Hint:* Prove by induction on k that $\mathbb{T}$ contains every subset with $n-k$ elements, where $0 \le k \le n-1$.]

(b) In the preceding, suppose we modify the assumptions so that X still contains at least $n+1$ points, but now every subset of X with exactly n points is a **closed** subset of X. Prove that $\mathbb{T}$ contains the cofinite topology.

3. Let $(X, \mathcal{U})$ andn $(Y, \mathcal{V})$ be topological spaces and let $f : X \to Y$ be a function which is not assumed to be continuous. If $\mathbf{S}$ is a subbase for $\mathcal{V}$ and the inverse image of every subset in $\mathbf{S}$ is open in $(X, \mathcal{U})$ show that f is in fact continuous.

4. Let (X, d) be a metric space and suppose $u, v \in X$ are such that $d(u, v) = 1$. Is there always a sequence of points $\{u_n\}$ in X such that $u_n \neq v$ for all n and $\lim_{n\to\infty} u_n = v$? Either prove this or give a counterexample.

5. Let $p, q \in \mathbb{R}^2$ and let $r, s > 0$ be real numbers. Show that the open neighborhoods $N_r(p)$ and $N_s(q)$ are homeomorphic by a map of the form $f(x) = mx + b$ where $b \in \mathbb{R}^2$ and $m > 0$. [*Hints:* Solve for b such that $f(p) = q$, and find m so that $|y - p| = \frac{1}{2}\, r$ implies $|f(y) - q| = \frac{1}{2}\, s$.]

2019 *problems*

1. Let $A \subset X$, and assume that X has the indiscrete topology. If $\mathcal{U}$ is the induced subspace topology on A, explain why $\mathcal{U}$ is equal to the indiscrete topology on A.

2. Show that the open intervals $(-\infty, b)$ and $(a, +\infty)$, where $a, b \in \mathbb{R}$, are a subbase for the standard metric topology on $\mathbb{R}$.

3. Let X be an infinite set with the cofinite topology, and let U be a nonempty open subset. Prove that the closure $\overline{U}$ is equal to X.

In each of the following problems, determine if the statement is always or sometimes true and sometimes false. In the second case, give examples where the statement is true and examples where it is false, using topological spaces with at least two points.

4. Let $(X, \mathcal{U})$ be a topological space, and let $\mathcal{U}'$ denote the sets of the form $X - W$ where $W \in \mathcal{U}$. Then $\mathcal{U}'$ defines a topology on X.

5. If $\mathcal{U}_1$ and $\mathcal{U}_2$ are topologies on a set X, then so is $\mathcal{U}_1 \cup \mathcal{U}_2$.

6. If X and Y are topological spaces with the cofinite topologies, then the product topology $X \times Y$ is the same as the cofinite topology on this set.

2017 *problems*

1. Define the concept of limit point for a subset of a topological or metric space.

2. Define what it means for a function f from one topological space $(X_1, \mathcal{T}_1)$ to another space $(X_2, \mathcal{T}_2)$ to be a homeomorphism.

3. Define the indiscrete topology on a set X.

4. Define the cofinite topology on a set X.

5. Define the concept of a base for a topological space $(X, \mathcal{T})$.

6. Let $\mathcal{A}$ be a family of subsets for some set X, and let $\mathcal{T}$ denote the topology on X generated by $\mathcal{A}$. Describe the elements of $\mathcal{T}$ in terms of unions and intersections of sets in $\mathcal{A}$.

7. Give an example of an open subset U in a topological space X such that U is properly contained in $\mathrm{Int}\,\overline{U}$. [*Hint:* There are simple examples where X is the real numbers with the usual topology.]

> *Note.* For every open subset $U \subset X$ we have $U \subset \mathrm{Int}\,\overline{U}$, and for many standard examples the two subsets are equal.

8. Give an example of a closed subset F in a topological space X such that F properly contains $\overline{\mathrm{Int}\,F}$. [*Hint:* There are simple examples where X is the real numbers with the usual topology.]

> *Note.* For every closed subset $F \subset X$ we have $F \supset \overline{\mathrm{Int}\,F}$, and for many standard examples the two subsets are equal.

9. Give examples of subsets $A, B \subset \mathbb{R}$ such that the intersection of the closures $\overline{A} \cap \overline{B}$ properly contains $\overline{A \cap B}$.

> *Note.* For every pair of subsets $A, B \subset X$ we know that $\overline{A} \cap \overline{B}$ contains $\overline{A \cap B}$, and in some cases equality holds.

10. Give examples of subsets $A, B \subset \mathbb{R}$ such that the union of the interiors $\mathrm{Int}\,A \cup \mathrm{Int}\,B$ is properly contained in $\mathrm{Int}\,A \cup B$.

> *Note.* For every pair of subsets $A, B \subset X$ we know that $\mathrm{Int}\,A \cup \mathrm{Int}\,B$ is contained in $\mathrm{Int}\,A \cup B$, and in some cases equality holds.

11. Give an example of an infinite sequence of closed subsets $F_n \subset \mathbb{R}$ such that the infinite union $\cup_n F_n$ is not closed.

12. Give an example of a pair of subsets $A, B \subset \mathbb{R}$ such that A is closed, B is open but not closed, and $A \cup B$ is closed.