Math 145A
Winter 2020

UPDATED GENERAL INFORMATION — MARCH 3, 2020

The third quiz

This will cover Chapters 7 – 11. Study problems are listed below. You may use Proposition C in your solution. Recall that a topological space is a $\mathbf{T}_1$ space if and only if all of its one point subsets are closed (and therefore all finite subsets are also closed).

New problems

Some of these are too challenging to be on a quiz, but such problems will be good practice exercises for the final examination.

1. (a) Let $f : X \to Y$ be a homeomorphism, and assume that X has the Hausdorff Separation Property. Prove that Y also has this property.

(b) Let $f : X \to Y$ be a homeomorphism, and assume that the topology on X comes from a metric d_X. Find a metric d_Y such that the topology on Y comes from d_Y.

2. (a) A topological space X is said to be *irreducible* if it cannot be written as the union of two proper closed subsets. Prove this equivalent to the condition that the intersection of two nonempty open sets is always nonempty.

(b) If X is an infinite space with the cofinite topology, explain why X is irreducible.

(c) If $f : X \to Y$ is a homeomorphism and X is irreducible, prove that Y is also irreducible.

(d) If X is a nonempty irreducible Hausdorff space, prove that X consists of a single point.

3. (a) Let $n \geq 2$ be an integer, and suppose that X is a topological space with more than n points such that every subset with exactly n elements is closed. Prove that X is a $\mathbf{T}_1$ space. [*Hint:* Prove by downward induction on $k \leq n$ that X contains every subset with exactly k elements.]

(b) Give a counterexample to the preceding conclusion when X has exactly n points.

4. Suppose that X and Y are topological spaces, and assume $A \subset X$ and $B \subset Y$ are nonempty subsets such that $A \times B$ is a closed subset of $X \times Y$ (with respect to the product topology). Show that A and B are closed in X and Y respectively. Also prove the analogous result when $A \times B$ is an open subset (*i.e.*, A and B are open in X and Y respectively). [*Hint:* The coordinate projections to X and Y are continuous and open, but they do **NOT** take closed subsets of the product to closed subsets of the factors. Consider vertical and horizontal slices $\{x\} \times Y$ and $X \times \{y\}$, and also their intersections with $A \times B$.]

5. Give an example to illustrate the assertion in the previous hint: The coordinate projections to X and Y do **NOT** necessarily take closed subsets of the product to closed subsets of the factors. [*Hint:* Consider the hyperbola in the coordinate plane defined by $xy = 1$.]

6. Suppose that X is a metric space and $A \subset X$.

(a) Show that $d(x, A) = \inf\{d(x, a)| \ a \in A$ is a continuous function of x. [*Hint:* Use $d(x, y) \geq |d(x, a) - d(y, a)|$.]

(b) Suppose that A is closed. Show that $d(x, A) = 0$ if and only if $x \in A$.

(c) Show that if A is closed then $A = \cap_n U_n$, a countable intersection of open sets. [*Hint:* Look at the sets where the distance is less than $1/n$ where n runs through the positive integers.]

(d) Give an example of a countable intersection of open sets (in some metric space) which is not a closed subset.

(e) Show that if X is an uncountable set with the cofinite topology, then the conclusion in (c) is not necessarily true.

7. If X is a topological space, prove that the following properties are equivalent:

(i) Every closed set is a countable intersection of open subsets.

(ii) Every open set is a countable union of closed subsets.

8. Let $f : X \to Y$ be a homeomorphism of topological spaces, and if $B \subset Z$ let $\mathbf{L}_Z(B)$ denote the set of limit points of B (in Z). Prove that

$$\mathbf{L}_Y(f[A]) \quad = \quad f\left[\mathbf{L}_X(A)\right] \ .$$

9. Let $f : X \to Y$ be a homeomorphism of topological spaces, and let $A \subset X$. Prove that f induces a homeomorphism h from $X - A$ to $Y - f[A]$ by the identity $h(x) = f(x$ for $x \in X - A$.

Old problems from 2019

1. Let $B \subset A \subset X$ where X is a topological space. Prove that if B is closed in X, then B is closed with respect to the subspace topology on A, and likewise if "open" replaces "closed." [*Hint:* How are A, B and $A \cap B$ related?]

2. Suppose that $A \subset X$ and $B \subset Y$ where X and Y are nonempty topological spaces, and assume that $A \times B$ is a nonempty open subset of $X \times Y$. Prove that A is open in X and B is open in Y, and likewise if "closed" replaces "open." [*Hint:* Take intersections with $\{x_0\} \times Y$ and $X \times \{y_0\}$ where $(x_0, y_0) \in A \times B$, and recall from the lectures that the vertical and horizontal slices are homeomorphic to the factors X and Y.]

3. Let X be a $\mathbf{T}_1$ space, let $A \subset X$, and let p be a limit point of A (no assumptions on whether or not $p \in A$). Show that every open subset V satisfying $p \in V$ must contain infinitely many points of A. [*Hint:* See the comments before the first problem.]

4. Prove that a Hausdorff space is a $\mathbf{T}_1$ space. [*Hint:* This was done in the lectures.]

5. Let $B \subset A \subset X$ where X is a topological space, and assume that (a) the subset A is closed in X, (b) the subset B is closed with respect to the subspace topology on A. Prove that B is closed in X, and likewise if "open" replaces "closed." [*Hint:* Use Proposition C or the definition of the subspace topology.]

6. Suppose that $A \subset X$ and $B \subset Y$ are closed. Show that $X \times B$, $A \times Y$ and $A \times B$ are closed in the product topology, and likewise if "open" replaces "closed." [*Hint:* The coordinate projections to X and Y are continuous and open, but they do **NOT** take closed subsets of the product to closed subsets of the factors.]