Math 145A
Winter 2020

UPDATED GENERAL INFORMATION — MARCH 5, 2020

Solutions to study problems for the third quiz

For the sake of convenience, statements of the problems are also included. A companion file `aabUpdate11a.145A.w20.pdf` contains illustrations for some problems. It is extremely likely that the third quiz will be one of these problems or a part of a problem.

1. (a) Let $f : X \to Y$ be a homeomorphism, and assume that X has the Hausdorff Separation Property. Prove that Y also has this property.

(b) Let $f : X \to Y$ be a homeomorphism, and assume that the topology on X comes from a metric d_X. Find a metric d_Y such that the topology on Y comes from d_Y.

SOLUTION

(a) Let p, q be distinct points in Y, and let $u, v \in X$ be the unique points such that $f(u) = p$ and $f(v) = q$. Then there are disjoint open subsets U, V in X containing u and v respectively. Since f is a homeomorphism, it takes open subsets to open subsets and disjoint subsets to disjoint subsets. Therefore $f[U]$ and $f[V]$ are disjoint open subsets containing $p = f(u)$ and $q = f(v)$ respectively.■

(b) Let $d_Y(y_1, y_2) = d_X(x_1, x_2)$ where $x_i = f^{-1}(y_i)$ for $i = 1, 2$. This is a metric because d_X is a metric. We claim that the induced metric topology on Y is equal to the original topology on Y.

First of all, each neighborhood $N^Y_\varepsilon(y)$ is open in the original topology because (i) its image under f^{-1} is just $N^X_\varepsilon(x)$, where $x = f^{-1}(y)$, (ii) the image of this open subset of X is an open subset in Y with respect to the original topology because f is a homeomorphism. Conversely, suppose that W is open with respect to the original topology on Y. Since f is a homeomorphism we know that $V = f^{-1}[W]$ is an open subset of the metric space X and hence for suitable choices of $\varepsilon(x) > 0$ we then have

$$V \;=\; \bigcup_{x \in V} N^X_{\varepsilon(x)}(x) \ .$$

Now let $y = f(x)$, so that $x = f^{-1})(y)$. Then it follows that $W = f[V]$ and

$$W \;=\; \bigcup_{y \in W} N^Y_{\varepsilon(x)}(y)$$

so that W is open with respect to the topology associated to the metric d_Y.■

2. (a) A topological space X is said to be *irreducible* if it cannot be written as the union of two proper closed subsets. Prove this equivalent to the condition that the intersection of two nonempty open sets is always nonempty.

(b) If X is an infinite space with the cofinite topology, explain why X is irreducible.

(c) If $f : X \to Y$ is a homeomorphism and X is irreducible, prove that Y is also irreducible.

(d) If X is a nonempty irreducible Hausdorff space, prove that X consists of a single point.

SOLUTION

(a) Suppose that X is irreducible and U, V are nonempty proper open subsets of X. Let $E = X - U$ and $F = X - V$. Then by the definition of an irreducible space we know that $E \cup F$ must be a proper subset of X and hence

$$X - (E \cup F) \quad = \quad (X - E) \ \cap \ (X - F) \quad = \quad U \ \cap \ V$$

must be nonempty. Conversely, if X satisfies the condition stated in the problem, suppose that E and F are proper closed subsets of X; we need to show that $E \cup F$ is also a proper subset of X. This is equivalent to saying that the complement is nonempty. Much as before, let $U = X - E$ and $V = X - F$. Then the displayed sequence of equations combines with the stated condition to show that $X - (E \cup F)$ is nonempty and hence $E \cup F$ must be a proper subset of X.■

(b) In the cofinite topology, the proper closed subsets are just the finite subsets of X. Since the union of two finite subsets is finite and the space is infinite, the union of two closed proper subsets can never be all of X.■

(c) Suppose that $Y = F_1 \cup F_2$ where F_1 and F_2 are closed subsets. If $E_i = f^{-1}[F_i]$ then it follows that $X = E_1 \cup E_2$ where E_1 and E_2 are closed subsets. Since X is irreducible, one of them, say E_1, must be equal to X. But f is 1–1 and onto, and this means that $F_1 = f[E_1] = f[X] = Y$. Therefore at least one of the two sets F_1, F_2 must be all of Y, and thus Y satisfies the condition for irreducibility.■

(d) We shall prove the contrapositive: *If X is Hausdorff and contains at least 2 points, then X is not irreducible.*

Assume the conditions in the contrapositive, and let p, q be distinct points in X. Then there are disjoint open subset U, V such that $p \in U$ and $q \in V$. Let $E = X - U$ and $F = X - V$. Then by construction $p \notin F$ and $q \notin E$, so that these closed subsets are proper. Combining these observations, we see that

$$E \cup F \quad = \quad (X - U) \ \cup \ (X - V) \quad = \quad X - (U \cap V) \quad = \quad X$$

(recall $U \cap V$ is empty) and therefore X is not irreducible.■

3. (a) Let $n \geq 2$ be an integer, and suppose that X is a topological space with more than n points such that every subset with exactly n elements is closed. Prove that X is a $\mathbf{T}_1$ space. [*Hint:* Prove by downward induction on $k \leq n$ that X contains every subset with exactly k elements.]

(b) Give a counterexample to the preceding conclusion when X has exactly n points.

SOLUTION

(a) Actually, a direct induction argument is probably cleaner. We must begin with the case $n = 2$, so that X contains at least 3 points. Suppose that $x \in X$. Since X has at least 3 points we can find a 3 point subset $\{x, y, z\} \subset X$. By the given assumption we know that both $\{x, y\}$ and $\{x, z\}$ are closed subsets. Therefore their intersection, which is merely $\{x\}$, must also be a closed subset, and it follows that X is a $\mathbf{T}_1$ space.

Assume now that the statement

If X has at least $k+1$ points and every subset with exactly k points is closed, then X is $\mathbf{T}_1$

is known for some $k = n \geq 2$; to complete an induction argument, we need to show that the result is also true for $k = n+1$. In this case the hypothesis means that each subset of X with exactly $n+1$ elements is closed in X, and to complete the inductive step it will suffice that X contains all subsets with exactly n elements; this is true because X has at least $n+2 > n+1$ elements, and hence the induction hypothesis (the statement is true for $k = n$) will imply that X is a $\mathbf{T}_1$ space.

So assume $A \subset X$ contains exactly n elements. Since X is assumed to contain at least $n+2$ points there are two points $p, q \in X - A$. We then have that

$$A \quad = \quad (A \cup \{p\}) \ \cap \ (A \cup \{q\})$$

and therefore A is an intersection of two closed subsets (each factor has $n+1$ points) and hence is also a closed subset. Therefore X satisfies the inductive condition for $k = n$, and by the induction hypothesis it follows that X must be a $\mathbf{T}_1$ space.■

(b) If X is a set with exactly $n \geq 2$ elements and is given the indiscrete topology, then there is only one subset with n elements — namely, X itself — and this subset is closed. However, this space is not a $\mathbf{T}_1$ space because one point subsets are not closed subsets of X.■

4. Suppose that X and Y are topological spaces, and assume $A \subset X$ and $B \subset Y$ are nonempty subsets such that $A \times B$ is a closed subset of $X \times Y$ (with respect to the product topology). Show that A and B are closed in X and Y respectively. Also prove the analogous result when $A \times B$ is an open subset (*i.e.*, A and B are open in X and Y respectively). [*Hint:* The coordinate projections to X and Y are continuous and open, but they do **NOT** take closed subsets of the product to closed subsets of the factors. Consider vertical and horizontal slices $\{x\} \times Y$ and $X \times \{y\}$, and also their intersections with $A \times B$.]

SOLUTION

Let $a \in A$ and $b \in B$, and consider the horizontal and vertical slice subspaces $X \times \{b\}$ and $\{a\} \times Y$ in $X \times Y$. In the lectures we showed that the maps $H_b : X \to X \times \{b\}$ and $V_a : Y \to \{a\} \times Y$ given by

$$H_b(x) \quad = \quad (x, b) \ , \qquad V_a(x) \quad = \quad (a, y)$$

are homeomorphisms. Under these homeomorphisms we have the following correspondences:

$$A \ \longleftrightarrow \ A \times \{b\} \quad = \quad (X \times \{b\}) \ \cap \ A \times B \ , \qquad B \ \longleftrightarrow \quad = \quad (\{a\} \times Y) \ \cap \ A \times B$$

(see Figure 1 in `aabNewUpdate11a.145A.w20.pdf` for a drawing).
Since $A \times B$ is a closed subset of $X \times Y$ it follows that $A \times \{b\}$ is a closed subset of $X \times \{b\}$ and $\{a\} \times Y$ is a closed subset of $X \times Y$. Since A corresponds to $A \times \{b\}$ under the homeomorphism from X to $X \times \{b\}$ and B corresponds to $\{a\} \times B$ under the homeomorphism from Y to $\{a\} \times Y$, it follows that A and B are closed in X and Y respectively.■

If we replace "closed" by "open" everywhere, the analogous reasoning remains valid, and therefore one obtains a similar conclusion with this substitution.■

5. Give an example to illustrate the assertion in the previous hint: The coordinate projections to X and Y do **NOT** necessarily take closed subsets of the product to closed subsets of the factors. [*Hint:* Consider the hyperbola in the coordinate plane defined by $xy = 1$.]

SOLUTION

The point of this exercise is that there is an alternate proof for the previous one in the case of open sets. Suppose that $A \times B$ is open in $X \times Y$, and let π_X, π_Y denote the coordinate projections onto X and Y respectively. We know that both of these mappings send open subsets to open subsets, and therefore $A = \pi_X[A \times B]$ and $B = \pi_Y[A \times B]$ are open subsets of X and Y respectively. *The same argument would work if these coordinate projection maps sent closed subsets to closed subsets, but it is important to recognize that coordinate projections do not necessarily have this property.*

Follow the hint and let $F \subset \mathbb{R} \times \mathbb{R}$ be the hyperbola defined by $xy = 1$. Since the function $m(x, y) = xy$ is continuous and one point subsets of a metric space are closed, it follows that F is closed. Now let p_1, p_2 denote the coordinate projections onto the first and second coordinates respectively. Then we have

$$\mathbb{R} - \{0\} \quad = \quad p_1[F] \quad = \quad p_2[F]$$

and this example has the desired properties since the left hand side is not an open subset of the real line.■

6. Suppose that X is a metric space and $A \subset X$.

(a) Show that $d(x, A) = \inf\{d(x, a)| \ a \in A$ is a continuous function of x. [*Hint:* Use $d(x, y) \geq |d(x, a) - d(y, a)|$.]

(b) Suppose that A is closed. Show that $d(x, A) = 0$ if and only if $x \in A$.

(c) Show that if A is closed then $A = \cap_n U_n$, a countable intersection of open sets. [*Hint:* Look at the sets where the distance is less than $1/n$ where n runs through the positive integers.]

(d) Give an example of a countable intersection of open sets (in some metric space) which is not a closed subset.

(e) Show that if X is an uncountable set with the cofinite topology, then the conclusion in (c) is not necessarily true.

SOLUTION

(a) By the Triangle Inequality we have $d(x, y) + d(x, a) \geq d(y, a)$ for all $x, y \in X$ and $a \in A$. Therefore if we take greatest lower bounds we have

$$d(x, y) \ + \ d(x, A) \ \geq \ d(y, A) \qquad \text{or eqivalently}$$

$$d(x, y) \ \geq \ d(y, A) \ - \ d(x, A)$$

for all x, y. If we switch the roles of x and y we obtain a similar inequality $d(x, y) \geq d(x, A) - d(y, A)$, and these combine to yield $d(x, y) \geq |d(x, A) - d(y, A)|$. Therefore if $d(x, y) < \varepsilon$ then we also have $|d(x, A) - d(y, A)| < \varepsilon$, and hence $d(x, A)$ is a continuous function of X.■

Notice that $d(x, A)$ is the greatest lower bound for a set of nonnegative real numbers and therefore must also be nonnegative.

(b) Suppose $x \in \overline{A}$; there are two cases depending upon whether $x \in A$ or $x \in \mathbf{L}(A)$. In the first case $0 \leq d(x, A) \leq d(x, x) = 0$ (recall $x \in A$, which means that $d(x, A) = 0$. In the second case, we know that for every $\varepsilon > 0$ there is some $a \in N_\varepsilon(x) \cap A$; therefore $\varepsilon > d(x, a) \geq d(x, A)$ for all $\varepsilon > 0$. This means that $0 \geq d(x, A) \geq 0$, so that $d(x, A) = 0$.

Conversely, suppose that $d(x, A) = 0$. Then for each $\varepsilon > 0$ there must be a point $a \in A$ so that $d(a, x) < \varepsilon$. The case $a = x$ is consistent with the conclusion we want, so assume $a \notin X$. Then we have

$$a \;\in\; (N_\varepsilon(x) - \{a\}) \;\cap\; A \;\neq\; \emptyset$$

for every $\varepsilon > 0$. Since this is the definition of a limit point, it follows that $x \in \overline{A}$.■

(c) Let $f_A(x) = d(x, A)$ and let $U_n = f_A^{-1}\,[\,[-\infty, 1/n)\,]$. Then each U_n is open in X, and since A is a closed subset we have $A = f^{-1}[\{0\}] = \cap_n U_n$ by part (b).■

(d) Let's see if we can find something which is relatively simple. One possibility is a half open interval such as $(0, 1]$. See Figure 2 for a drawing which corresponds to the construction given below.

Consider the open intervals in the real line given by

$$U_n \;=\; \left(0, 1 + \frac{1}{n}\right)$$

where n runs through all the positive integers. Then $\cap_n U_n = (0, 1]$.■

7. If X is a topological space, prove that the following properties are equivalent:

(i) Every closed set is a countable intersection of open subsets.

(ii) Every open set is a countable union of closed subsets.

SOLUTION

$(i) \Longrightarrow (ii)$ Let U be an open subset, and let $F = X - U$. Then $F = \cap_n V_n$ where the sets V_n are open in X, and therefore

$$U \;=\; X - F \;=\; \bigcup_n X - V_n$$

where each of the sets $X - V_n$ is closed.

$(ii) \Longrightarrow (i)$ Let F be a closed subset, and let $U = X - F$. Then $U = \cap_n E_n$ where the sets E_n are closed in X, and therefore

$$F \;=\; X - U \;=\; \bigcap_n X - E_n$$

where each of the sets $X - E_n$ is open.■

8. Let $f : X \to Y$ be a homeomorphism of topological spaces, and if $B \subset Z$ let $\mathbf{L}_Z(B)$ denote the set of limit points of B (in Z). Prove that

$$\mathbf{L}_Y(f[A]) \;=\; f\,[\mathbf{L}_X(A)]\;.$$

SOLUTION

Suppose that $y \in \mathbf{L}_Y(f[A])$. then for every open set V such that $y \in V$ we have $(V - \{y\}) \cap f[A] \neq \emptyset$. Let $x \in X$ be the unique point such that $f(x) = y$; we want to show that x is a limit point of A. Let U be an open subset of X with $x \in U$, and let $W = f[U]$. Then W is an open set containing

$y = f(x)$ and hence $(W - \{f(x)\}) \cap f[A] \neq \emptyset$. Let $u = f(z)$ be a point in this intersection (we can write the point as $f(z)$ for some z because f is onto). Then it follows that $z \in (U - \{x\}) \cap A$ because f is 1–1. Since U was an arbitrary open subset containing x, it follows that $x \in \mathbf{L}_X(A)$ and hence $y = f(x) \in f\,[\mathbf{L}_X(A)]$.

Conversely, suppose that $y = f(x) \in f\,[\mathbf{L}_X(A)]$; we want to show that y is a limit point of $f[A]$. Let V be an open set containing y, and again choose U so that $f[U] = V$. By construction U is an open set containing x, and by construction x is a limit point of A. Therefore there is some point $z \neq x$ such that $z \in U \cap A$. Since $z \neq x$ it follows that $f(z) \neq f(x) = y$ and $f(z) \in f[U] \cap f[A] = V \cap f[A]$. Since V was arbitrary, it follows that $y \in \mathbf{L}_Y(f[A])$.■

Strong recommendation. Draw some pictures of X and Y while studying this proof.

9. Let $f : X \to Y$ be a homeomorphism of topological spaces, and let $A \subset X$. Prove that f induces a homeomorphism h from $X - A$ to $Y - f[A]$ by the identity $h(x) = f(x)$ for $x \in X - A$.

SOLUTION

Since f is 1–1 onto, the same is true of h, and it suffices to show that a set $V \subset X - A$ is open if and only if $f[V] \subset Y - f[A]$ is open in the subspace topology.

Suppose that $V \subset X - A$ is open in the subspace topology, so that $V = (X - A) \cap W$ where W is open in X. Since f is 1–1 onto we have $f[V] = f[X - A] \cap f[W] = (Y - f[A]) \cap f[W]$. Now f sends open subsets in X to open subsets in Y (it is a homeomorphism), so it follows that $h[V] = f[V]$ is an open subset of $Y - f[A]$ in the subspace topology for the latter.

The other direction is similar. Suppose that $U \subset Y - f[A]$ is open in the subspace topology, so that $U = (Y - f[A]) \cap \Omega$ where Ω is open in Y. If $g = f^{-1}, then g[U] = g(Y - f[A]) \cap g[\Omega]$ as before, and since g and f are inverses of each other the right side of the equation reduce to $g[U] = (X - A) \cap g[\Omega]$, which is open with respect to the subspace topology on $X - A$.■

Old problems from 2019

1. Let $B \subset A \subset X$ where X is a topological space. Prove that if B is closed in X, then B is closed with respect to the subspace topology on A, and likewise if "open" replaces "closed." [*Hint:* How are A, B and $A \cap B$ related?]

SOLUTION

If B is closed in X, then $B = B \cap A$ because $B \subset A$ and hence B is closed with respect to the subspace topology on A. The same argument is valid if "open" replaces "closed."■

2. Suppose that $A \subset X$ and $B \subset Y$ where X and Y are nonempty topological spaces, and assume that $A \times B$ is a nonempty open subset of $X \times Y$. Prove that A is open in X and B is open in Y, and likewise if "closed" replaces "open." [*Hint:* Take intersections with $\{x_0\} \times Y$ and $X \times \{y_0\}$ where $(x_0, y_0) \in A \times B$, and recall from the lectures that the vertical and horizontal slices are homeomorphic to the factors X and Y.]

This is a duplication of a problem from the first part.

3. Let X be a $\mathbf{T}_1$ space, let $A \subset X$, and let p be a limit point of A (no assumptions on whether or not $p \in A$). Show that every open subset V satisfying $p \in V$ must contain infinitely many points of A. [*Hint:* See the comments before the first problem.]

SOLUTION

Proceed by induction on $n \geq 1$ to show that $(V - \{p\}) \cap A$ contains at least n points. The case $n = 1$ is just the definition of limit point.

We need to show that if the statement is true for n, then it is also true for $n+1$. Using the inductive assumption, let E be a subset of $(V - \{p\}) \cap A$ consisting of n points. Since X is $\mathbf{T}_1$ this finite set is closed, so that $U = V - E = (X - E) \cap V$ is also an open subset, and by construction it contains p. Therefore the intersection $(U - \{p\}) \cap A$ is nonempty and contains some point q, so that $E \cup \{q\}$ is a subset of $(V - \{p\}) \cap A$ consisting of $n + 1$ points. This completes the proof of the inductive step.

The preceding shows that $(V - \{p\}) \cap A$ contains at least n points for every positive integer n, and the latter implies that this intersection must be infinite.■

4. Prove that a Hausdorff space is a $\mathbf{T}_1$ space. [*Hint:* This was done in the lectures.]

SOLUTION

Suppose that $x \in X$ and let $y \neq x$. Then there are disjoint open subsets U_y and V_y such that $x \in U_y$ and $y \in V_y$. It now follows that

$$X - \{x\} \quad = \quad \bigcup_{y \neq x} \{y\} \quad \subset \quad \bigcup_{y \neq x} V_y \quad \subset \quad X - \{x\}$$

so that the last two subsets in this display must be equal. But this means $X = \{x\}$ is open, which is the same as saying that $\{x\}$ is closed.■

5. Let $B \subset A \subset X$ where X is a topological space, and assume that (a) the subset A is closed in X, (b) the subset B is closed with respect to the subspace topology on A. Prove that B is closed in X, and likewise if "open" replaces "closed." [*Hint:* Use Proposition C in a previous update file or the definition of the subspace topology.]

SOLUTION

By Proposition C we know that $B = A \cap C$ where C is a closed subset of X. Therefore B is an intersection of two closed subsets and hence is closed. The same argument is valid if "open" replaces "closed."■

6. Suppose that $A \subset X$ and $B \subset Y$ are closed. Show that $X \times B$, $A \times Y$ and $A \times B$ are closed in the product topology, and likewise if "open" replaces "closed." [*Hint:* The coordinate projections to X and Y are continuous and open, but they do **NOT** take closed subsets of the product to closed subsets of the factors.]

SOLUTION

Let π_X, π_Y denote the coordinate projections onto X and Y respectively. Then $X \times B = \pi_X^{-1}[B]$ is closed because π_X is continuous and $B \subset Y$ is closed. Similarly $A \times Y = \pi_Y^{-1}[A]$ is closed because π_X is continuous and $A \subset X$ is closed. Therefore the intersection

$$A \times B \quad = \quad (A \times Y) \ \cup \ (X \times B)$$

is also closed.

The same argument is valid if "open" replaces"closed." However, in this case one can simply use the definition of the product topology to verify this statement.■