Math 145A
Winter 2020

UPDATED GENERAL INFORMATION — MARCH 12, 2020

Additional hints for practice problems

The first part of this document contains fairly broad hints for the 2020 problems. A file with solutions and hints for problems from previous years is also attached.

1. Let $F = \{y\}$ where $y \neq x$, and consider the inverse images of $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$ with respect to f. Why are these subsets open and disjoint?

2. Strictly speaking we should restrict attention to subbases $\mathcal{A}$ such that the entire space X is a union of finite intersections of sets in $\mathcal{A}$ and $\mathcal{A}$ contains the empty set.

By definition (and the preceding stipulation), if $\mathcal{A}$ is a subbase for the topology, then every open subset is a union of finite intersections of sets in $\mathcal{A}$. Specializing to the data of the problem, we know that if U is an open subset then U is a union of finite intersections of sets in $\mathcal{V}$, and likewise for $\mathcal{W}$. Since $U \cap U = U$, if we take the intersections of these two descriptions, why does this form a third description involving intersections of sets in $\mathcal{V}$ and $\mathcal{W}$?

3. The only complicated part of verifying that d^* defines a metric is checking the Triangle Inequality; all the others follow directly from the definition of d^* in terms of the original metric d. Proving that $d^*(x,z) \leq d^*(x,y) + d^*(x,z)$ can be split into 8 cases depending upon whether $d(u,v) \leq 1$ or $d(u,v) > 1$ for each of the pairs $(u,v) = (x,z),\ (x,y),\ (y,z)$. To complete the problem, consider the hint already given and explain why $N^d_\varepsilon(x) = N^{d'}_\varepsilon(x)$ for $\varepsilon < 1$.

4. Prove by induction on k that the set $U_0 \cap U_1 \cap \cdots U_k$ is connected when $k \leq n$.

5. If a is a limit point of A and b is a limit point of B, why is (a,b) a limit point of $A \times B$? For the final part, what happens if Y is a single point?

6. What happens if the endpoints are irrational numbers? Consider the set of all such intervals such that the midpoint is a rational number. Why doe these form a base for the topology?

7. Why is each $U_\alpha - \{x\}$ open in both U_α and X? There are two cases depending on whether or noot $x \in U_\alpha$. Why is $X - \{x\}$ the union of the sets $U_\alpha - \{x\}$?

Remark. The analogous statement with "Hausdorff" replacing $\mathbf{T_1}$ turns out to be false.

8. Recall that if $x_0 \in X$ then $f(x) = d(x, x_0)$ is a nonconstant continuous real valued function of X. Since X is infinite, the image is connected and contains more than one point. What does this imply about the cardinal number of X?

There is no Problem 9.

10. (a) Once again, if $g(y) = d(x, y)$, then $g : X \to \mathbb{R}$ is continuous. By definition the set $CN_\varepsilon(x)$ is the inverse image of the closed interval $[0, 1]$ under g, so by continuity this subset is closed in X.

(b) Let $X_1 = [-\varepsilon, \varepsilon]$, $X_2 = \{-\varepsilon, 0, \varepsilon\}$ and $X_3 = [-\varepsilon, 0] \cup \{\varepsilon\}$. Then in X_1 we have

$$(-\varepsilon, \varepsilon) = N_\varepsilon(0) \neq \overline{N_\varepsilon(0)} = CN_\varepsilon(0) = [-\varepsilon, \varepsilon]$$

in X_2 we have

$$\{0\} = N_\varepsilon(0) = \overline{N_\varepsilon(0)} \neq CN_\varepsilon(0) = \{-\varepsilon, 0, \varepsilon\}$$

and in X_3 we have

$$(-\varepsilon, 0] = N_\varepsilon(0) \ , \quad [-\varepsilon, 0] = \overline{N_\varepsilon(0)} \ , \quad CN_\varepsilon(0) = [-\varepsilon, 0] \cup \{\varepsilon\}$$

so that $N_\varepsilon(0)$ is properly contained in $\overline{N_\varepsilon(0)}$ and the latter is propely contained in $X_3 = CN_\varepsilon(0)$.

11. Let d and d' be the discrete and the standard metric on the real line. Why is the identity mapping $id_X : (X, d') \to (X, d)$ not continuous? To form the desired infinite family $\{A_\alpha\}$, consider all singleton sets $\{x\}$ where x runs through all the elements of X.