UPDATED GENERAL INFORMATION — JANUARY 28, 2019

Solutions to Quiz 1 *practice problems*

1. Let X be the set of all sequences $a = \{a_n\}$ taking values in $[0,1]$, and define

$$d(a,b) \quad = \quad \sum_{n=1}^{\infty} \frac{|b_n - a_n|}{2^n} \ .$$

Show that the infinite series on the right hand side always converges and the formula defines a metric on X.

SOLUTION

Since $0 \leq a_n, b_n \leq 1$ we have $|a_n - b_n| \leq 2$ and therefore we have

$$\sum_{n=1}^{\infty} \frac{|b_n - a_n|}{2^n} \quad \leq \quad \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \ .$$

The right hand side coverges, so by the comparison test the left hand side does too.

We now need to show that the function defines a metric on X. Since the infinite sum consists of nonnegative terms, we know that $d(a,b) \geq 0$. Furthermore, if equality holds then all the summands must be zero, which is equivalent to saying that $a_n = b_n$ for all n. The symmetry property of the metric follows because $|a_n - b_n| = |b_n - a_n|$ for all n. Finally, the Triangle inequality holds because $|a_n - b_n| = |(a_n - c_n) - (b_n - c_n)| \leq |a_n - c_n| + |c_n - b_n|$ for all n.■

2. Let X be the set of all polynomials over the real numbers, and define

$$d(p,q) \quad = \quad \int_0^1 \ |p(t) - q(t)| \, dt \ .$$

Prove that this formula defines a metric on X. [*Hint:* If the right hand side is zero, why do we have $p(t) = q(t)$ for all $t \in [0,1]$, and why does this imply that $p(t) = q(t)$ everywhere? Recall that polynomial functions are continuous.]

SOLUTION

Since the integrand of the expression on the right is nonnegative, it follows that the integral is also nonnegative. It is clearly zero if $p = q$. Suppose now that it is zero for some p and q. Now $|p - q|$ is a polynomial (hence continuous) function, and either $p = q$ or else there are only finitely many real numbers r such that $p(r) - q(r) = 0$. In the latter case, there is some $c \in [0,1]$ for which the difference is nonzero, and by continuity there is some interval $[u,v] \subset [0,1]$ containing c such that $|p - q| > h$ on $[u,v]$ for some $h > 0$. Therefore we have

$$0 \quad < \quad h(v-u) \quad < \quad \int_u^v \ |p(t) - q(t)| \, dt \quad \leq \quad \int_0^1 \ |p(t) - q(t)| \, dt$$

so that the right hand side is positive if $p \neq q$.

The remaining conditions are much easier to verify. In particular, $d(p,q) = d(q,p)$ because $|p-q| = |q-p|$, and the triangle inequality follows because $0 \le |p-s| \le |p-q| + |q-s|$ for all polynomials p, q, s; this inequality implies a corresponding inequality for integrals over $[0,1]$.■

3. Suppose that (X,d) is a metric space such that $d(u,v) < \pi/4$ for all u and v. Prove that $\sin d(u,v)$ defines a metric on X. [*Hint:* Use trigonometric identities to show that $\sin(\alpha+\beta) \le \sin\alpha + \sin\beta$ for $0 \le \alpha, \beta \le \pi/4$.]

SOLUTION

Let's begin with the hint. We know that

$$\sin(\alpha+\beta) \quad = \quad \sin\alpha\cos\beta \; + \; \sin\beta\cos\alpha$$

and since $0 \le \alpha, \beta \le \pi/4$ we know that each of the sines and cosines lies in $[0,1]$. Therefore the right hand side is less than or equal to $\sin\alpha + \sin\beta$.

We shall now verify that $\sin d(u,v)$ defines a metric on X. Since $d(u,v) < \pi/4$ it follows that $\sin d(u,v) \ge 0$, and if equality holds then $d(u,v) = 0$; since d is a metric this means that $u = v$. Furthermore, $d(u,v) = d(v,u)$ by the symmetry property of distances, and therefore we also have $\sin d(u,v) = \sin d(v,u)$. Finally, we need to verify the Triangle Inequality. Since d is a metric, the hypotheses imply that $d(u,v) \le d(u,w) + d(w,v) \le \pi/2$ for all $u, v, w \in X$, and since the sine function is increasing on $[0, \pi/2]$ we have $\sin d(u,v) \le \sin\left(d(u,w) + d(w,v)\right)$. By the observation in the first paragraph the right hand side is less than or equal to $\sin d(u,w) \; + \; \sin d(w,v)$, and if we combine this with the preceding sentence we obtain the Triangle Inequality for the function $\sin d(u,v)$.■

These are more complicated than quiz questions, but they illustrate the general pattern for determining whether a function $f(x_1, x_2)$ defines a metric; namely, one has to show that each of the properties in the definition is satisfied in order to verify that one has a metric space. If any of these properties is false for some specific choices of points in X, then the function does not define a metric.