Comments on practice problems in

aabUpdate04.145A.w19.pdf

New problems

1. Let $X = \{x_1, \ldots, x_n\}$. If $x_i \in X$ let $a_i = \min_{j \neq i} d(x_j, x_i)$, so $a_i > 0$. Take $r = \frac{1}{2} a_i$. Then $N_r(x_i) = \{x_i\}$.

2. A open, closed in X.
A open in $X \Rightarrow X - A$ closed.
A closed in $X \Rightarrow X - A$ open.

3. Let $X = \mathbb{R}$, and take $A_n = \{\frac{1}{n}\}$, which is closed, so $\cup A_n = \cup \overline{A_n}$. However, $0 \in \overline{\cup A_n}$.
Now $\overline{\cup A_n}$ is closed and contains A_k for all k, so $\overline{\cup A_n} \supseteq \overline{A_k}$ all k and hence $\overline{\cup A_n} \supseteq \cup \overline{A_n}$. One example for the last sentence is $A_n = \{n\}$, $n \in \mathbb{N}$.

4. (a) Let $x \in U$, let e be a unit vector, and choose $r > 0$ so $N_r(x) \subseteq U$. Then x is the limit of the sequence $x + \frac{r}{n} e$, and none of these $= x$.

(b) By #1 each $\{x\}$ is open in $X \Rightarrow$ $\{x\} = N_r(x)$ some $r > 0$. Hence $X \cap (N_r(x) - \{x\}) = X \cap \phi = \phi$ and by def x is not a limit point.

(c) Take $\mathbb{Q}^n \subseteq \mathbb{R}^n$.

5. Imitate the case of a twofold product. By construction $d_i(q_i(u), q_i(v)) \leq D(u,v)$ so $q_i : X_1 \times X_2 \times X_3 \longrightarrow X_i$ is continuous (given ε, take $\delta = \varepsilon$). Hence f cont. $\Rightarrow$ each $q_i f$ cont. Conversely, if each $q_i f = f_i$ cont and $\varepsilon > 0$ and $y_0 \in Y$ chose $\delta_i > 0$ so

$$d_Y(y_0, y) < \delta_i \Rightarrow d(f_i(y_0), f_i(y)) < \frac{\varepsilon}{\sqrt{3}}$$

Let $\delta = \min \delta_i$. Then $d(y, y_0) < \delta \Rightarrow$
$D(f(y), f(y_0)) < \varepsilon$.

6. Use #5 and note that

$$q_1 F = q_2, \quad q_2 F = q_3, \quad q_3 F = q_1.$$

<u>Old problems</u>

1. The containment follows because $b \in B \Rightarrow f(b) \in f[B]$, so that $b \in f^{-1}[f[B]]$.
For the example with proper containment, take $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$, $B = [0, 1]$.
Then the right hand side is $[-1, 1]$.

2. PASS

3. By definition we have $f(b) < f(c_+) + \varepsilon$ for some $b > c$ and $f(a) > f(c_-) - \varepsilon$ for some $a < c$. Hence

$$f(c_-) - \varepsilon < f(x) < f(c_+) + \varepsilon$$

if $a < x < b$.

If $f(c-) = f(c+)$, then f is cont. at c;
take δ so $(c-\delta, c+\delta) \subseteq (a,b)$.

If $f(c_-) < f(c_+)$ there are sequences
$L_n \to c$ $(L_n \leq c)$ $U_n \to c$ $(U_n \geq c)$
with $\lim f(L_n) = f(c-)$, $f(U_n) = f(c_+)$
and $f(c-) \neq f(c+) \Rightarrow f$ not continuous at c.

4 (i) A: $0\,(\quad\quad)\,1$

B: $\tfrac{1}{2}\,(\quad\quad]\,\tfrac{3}{2}$

(ii) A: $0\,[\quad\quad]\,1$

B: $\tfrac{1}{2}\,[\quad\quad)\,\tfrac{3}{2}$

(iii) A: $0\,(\quad\quad]\,1$

B: $\tfrac{1}{2}\,[\quad\quad)\,\tfrac{3}{2}$

(iv) A: $0\,[\quad\quad)\,\tfrac{2}{3}$

B: $\tfrac{1}{3}\,(\quad\quad]\,1$

5. Suppose $b \in \bar{A}$. Then if $\varepsilon > 0$ can find $a \in A$ so $b - a \leq \varepsilon$. Hence

$$(N_\varepsilon(b) - \{a\}) \cap A \neq \phi$$ because $b \notin A$.

6. (i) A open $\Rightarrow A = X \cap A$ & X is closed
A closed $\Rightarrow A = X \cap A$ & X is open

(ii) $$[a, b) = [a, b+1] \cap (a-1, b)$$
$$(a, b] = (a, b+1) \cap [a-1, b].$$

(iii) Follow the hint. $Q = E \cap V \Rightarrow Q \subseteq E$, E closed. But then $E = \mathbb{R}$, so $Q = \mathbb{R} \cap V = V$, V open. Since V is not open this is impossible.

7. PASS

8. (i) $\sqrt{u+v} \leq \sqrt{u} + \sqrt{v}$ because

$(\sqrt{u+v})^2 = u + v$

$(\sqrt{u} + \sqrt{v})^2 = u + 2\sqrt{uv} + v$

and the first expression $\leq$ second.

(ii) The only nontrivial point to check is the Triangle Inequality. Do this verification using (i).

(iii) PASS

9. $d^* = \max d, d' \geq 0$ because $d, d' \geq 0$

If $d^* = 0$ then $\max d, d' = 0 \Rightarrow$ $d(u,v) = 0$ or $d'(u,v) = 0$. In either case $u = v$.

$d^*(u,v) = d^*(v,u)$ because $d^* = \max d, d'$ and both of the latter have this property.

Finally

$d^*(x,z) = \max d(x,z), d'(x,z)$.

Say the max is $d(x,z)$.

Then $d^*(x,z) = d(x,z) \le d(x,y) + d(x,z)$

$\le d^*(x,y) + d^*(y,z)$ since $d \le d^*$.

10. (i) Let $M = \max |f'|$ on $[0,1]$.

$0 < u < v < 1$. Then by the MVT

$|f(u) - f(v)| = |f'(W)\cdot(u-v)|$ where

$u < W < v$. The RHS $\le M \cdot |u-v|$.

(ii) If $d(x_1, x_2) < \frac{\varepsilon}{K}$ then

$$d(f(x_1), f(x_2)) \le K \cdot d(x_1, x_2) <$$

$$K \cdot \frac{\varepsilon}{K} = \varepsilon.$$