Math 145A
Winter 2019

UPDATED GENERAL INFORMATION — MARCH 1, 2019

Alternate characterization of the subspace topology

The proof of the following result was never properly finished in the lectures, so here is a complete argument:

PROPOSITION C. *Let $B \subset A \subset X$ where X is a topological space. Then B is closed with respect to the subspace topology on A if and only if $B = A \cap F$ where F is a closed subset of X.*

Proof. B is closed in A if and only if $A - B$ is open in A, which by definition is true if and only if $A - B = A \cap V$ where V is open in X.

Regardless of whether V is open in X, if $A - B = X \cap V$ we have

$$B \;=\; A - (A - B) \;=\; A - (X \cap V) \;=\; (X - V) \cap A \;.$$

Combining the preceding two observations, we see that B is closed in A if and only if $B = (X - V) \cap A$ for some open subset $V \subset X$, which is equivalent to saying that $B = F \cap A$ where F is a closed subset of X.■

The third quiz

This will be taken from the six problems listed below. You may use Proposition C in your solution. Recall that a topological space is a $\mathbf{T}_1$ space if and only if all of its one point subsets are closed (and therefore all finite subsets are also closed).

1. Let $B \subset A \subset X$ where X is a topological space. Prove that if B is closed in X, then B is closed with respect to the subspace topology on A, and likewise if "open" replaces "closed." [*Hint:* How are A, B and $A \cap B$ related?]

2. Suppose that $A \subset X$ and $B \subset Y$ where X and Y are nonempty topological spaces, and assume that $A \times B$ is a nonempty open subset of $X \times Y$. Prove that A is open in X and B is open in Y, and likewise if "closed" replaces "open." [*Hint:* Take intersections with $\{x_0\} \times Y$ and $X \times \{y_0\}$ where $(x_0, y_0) \in A \times B$, and recall from the lectures that the vertical and horizontal slices are homeomorphic to the factors X and Y.]

3. Let X be a $\mathbf{T}_1$ space, let $A \subset X$, and let p be a limit point of A (no assumptions on whether or not $p \in A$). Show that every open subset V satisfying $p \in V$ must contain infinitely many points of A. [*Hint:* See the comments before the first problem.]

4. Prove that a Hausdorff space is a $\mathbf{T}_1$ space. [*Hint:* This was done in the lectures.]

5. Let $B \subset A \subset X$ where X is a topological space, and assume that (a) the subset A is closed in X, (b) the subset B is closed with respect to the subspace topology on A. Prove that B is closed in X, and likewise if "open" replaces "closed." [*Hint:* Use Proposition C or the definition of the subspace topology.]

6. Suppose that $A \subset X$ and $B \subset Y$ are closed. Show that $X \times B$, $A \times Y$ and $A \times B$ is closed in the product topology, and likewise if "open" replaces "closed." [*Hint:* The coordinate projections to X and Y are continuous and open, but they do **NOT** take closed subsets of the product to closed subsets of the factors.]

Assignments for Chapters $11 - 13$

Working the exercises listed below is **strongly recommended.**

The following exercises are taken from Sutherland:

- Chapter 11: 11.1, 11.3, 11.6, 11.8
- Chapter 12: 12.1(i) – (iii), 12.3, 12.7, 12.13, 12.15, 12.17
- Chapter 13: 13.1–4, 13.10. 13.14, 13.20

The following references are to the file `exercises05w14.pdf` in the course directory.

- Additional exercise for Chapter 11: 1 – 3
- Additional exercises for Chapter 12: 1, 2(i), 4, 5
- Additional exercises for Chapter 13: 1, 2, 3(i), 4

Reading assignments from solutions to exercises

Two additional strong recommendations are to read through the solutions to Exercises 12.9, 12.11, 12.16, 13.9, and 13.13 from Sutherland.