Math 145A
Winter 2019

UPDATED GENERAL INFORMATION — MARCH 11, 2019

The final examination

This examination will take place on Monday, March 18, from 3 to 6 P.M. It will consist of six questions, and it is designed to be 50 per cent longer than the midterm examination; however, you will have the entire three hour period to complete the examination. Roughly 80 per cent of the problems will involve material since the midterm (starting with `math145Anotes06c.pdf`. Most of the problems will be explanations or proofs, and the others will involve examples; the former are likely to include questions about how the abstract definitions apply to subsets of the real line or coordinate plane.

As usual, the examination will involve knowledge of definitions, basic results, their meanings in simple cases, some relatively uncomplicated logical derivations or proofs, and applications to specific questions which arise in single variable calculus. The assigned questions in the exercises, the contents of the examinations from the Winter 2014, 2016 and 2017 courses. Here are some further items for study:

New problems

1. Let $0 \neq m \in \mathbb{R}$, and consider the line $L \subset \mathbb{R}^2$ defined by $y = mx$.

(a) Prove that the sets of points U_+ and U_- defined by $y > mx$ and $y < mx$ are open and connected (in fact, they are convex).

(b) If $\gamma(t)$ is a parametrized curve in the coordinate plane (where $t \in [-1, 1]$) such that $\gamma(-1) \in U_-$ and $\gamma(1) \in U_+$, explain why there is some t_0 such that $\gamma(t_0) \in L$.

(c) Prove that every point of L is a limit point of both U_+ and U_-.

2. Prove that a product of $\mathbf{T}_1$ spaces is also a $\mathbf{T}_1$ space, and that a subspace of $\mathbf{T}_1$ spaces is also a $\mathbf{T}_1$ space.

3. Suppose that the space X is a union of the closed subspaces $A \cup B$. Prove that X is Hausdorff if both A and B are. [*Comments:* Recall that a space is Hausdorff if and only if the diagonal is closed in the product of the space with itself. It turns out that the analog of this statement is false if A and B are open instead of closed.]

3. Let $h : X \to Y$ be a homeomorphism, and let $A \subset X$. Prove the identity $h[L_X(A)] = L_Y(h[A])$, where $L_Z(B)$ denotes the set of limit points for a subset B in a space Z.

4. Let $h : X \to Y$ be a homeomorphism, and let C be a connected component in X. Prove that $h[C]$ is a connected component in Y. Also state and prove a corresponding result for arc components.

5. Prove that a finite union of compact subspaces (in some space X) is compact, and give an example to show this fails for infinite unions.

6. (*a*) Suppose that the space X is a union of two open subsets U and V, and let $f : X \to Y$ be a set-theoretic map from X to a topological space Y such that the restrictions $f|U$ and $f|V$ are both continuous. Prove that f is continuous.

(*b*) Does a similar result hold if *closed* replaces *open*? Either prove this or give a counterexample.

7. Show that value of the polynomial function

$$p(x,y) \quad = \quad x^5 \; - \; 2x^4y \; + \; 3x^3y^3 \; - \; 4x^2y^3 \; + \; 5x^4y \; - \; 6y^5$$

can be any number between 1 and -1 at some point(s) of the solid square $[0,1] \times [0,1]$. [*Hint:* What are its values at the corner points of the square?]

8. Let X be a metric space, let $f : X \to \mathbb{R}$ be continuous, and let $A \subset X$ be compact. Prove that the restriction of f to the set $L_X(A)$ of limit points of A has a maximum value.

Older problems 1

1. Suppose that X is a connected space which is the union of connected subspaces $A_1, \; \cdot \;, A_n$, where for each $i \geq 2$ we have $A_i \cap A_{i-1} \neq \emptyset$. Prove that $\cup_{i \leq n} A_i$ is connected. [*Hint:* For each k prove by induction that that $\cup_{i \leq k} A_i$ is connected.]

2. (*i*) A topological space X is said to have the Kolmogorov separation property if for each pair of distinct points in X there is an open subset containing one but not the other. Prove that if X and Y have the Kolmogorov separation property then so does their product $X \times Y$ (with the product topology).

(*ii*) Suppose that X and Y are two topological spaces in which each one point subsets is closed (the Frechet separation property). Prove that their product $X \times Y$ (with the product topology) also has this property.

3. (*i*) Suppose that X is a topological space and A is a subset which is not closed. Explain why A is a proper subset of its closure (in X).

(*ii*) Suppose that X is a connected topological space and U is a nonempty proper open subset of X. Prove that U is a proper subset of its closure (in X).

4. (*i*) Let $U \subset \mathbb{R}^2$ be the open first quadrant defined by $x, y > 0$. Find the set of limit points for U, and prove that your answer is correct.

(*ii*) Let $\mathcal{V}$ be the topology on $\mathbb{R}$ whose nonempty proper open subsets are the open rays $(c, +\infty)$ where $c \in \mathbb{R}$. Show that the rational numbers are dense in the topological space $(\mathbb{R}, \mathcal{V})$.

(*iii*) Explain why the topological space in (*ii*) is connected.

5. (*i*) A subset A of a topological space X is said to be locally closed if it is the intersection of a closed set and an open set. Prove that if A and B are locally closed in X, then so is their intersection $A \cap B$.

(*ii*) Suppose that X and Y are topological spaces such that $\mathcal{A} = \{U_\alpha\}$ and $\mathcal{B} = \{V_\beta\}$ are bases for the topologies on X and Y respectively. Prove that the set of all products $U_\alpha \times V_\beta$ is a base for the product topology on $X \times Y$.

(*iii*) Show that the topological space in 4(*ii*) satisfies the Kolmogorov separation property but does not satisfy the $\mathbf{T}_1$ property.

6. Let $X = \mathbb{R}^2$, and let A be the set of points (x, y) in X such that either $y = 0$ or $x > 0$. Find the interior and boundary of A.

Older problems 2

(1) Suppose that X is a topological space and every nonempty open subset $U \subset X$ contains at least two points. Prove that every point of X is a limit point of X.

(2) Let $A \subset X$ and $B \subset Y$, where X and Y are topological spaces. Prove that the limit point sets of A, B and $A \times B$ satisfy $L(A) \times L(B) \subset L(A \times B)$. As usual, take the product topology on $X \times Y$.

(3) Let A, B, X, Y be as in the previous exercise where both X and Y are nonempty, and assume that $A \times B$ is an open subset of $X \times Y$ with respect to the product topology. Prove that A and B are open in X and Y respectively. [*Hints:* Recall that the vertical and horizontal slices $\{x_0\} \times Y$ and $X \times \{y_0\}$ are homeomorphic to Y and X respectively by the maps sending (x_0, y) and (x, y_0) to y and x. Also, recall the definition of subspace topologies for vertical and horizontal slices.] — Why is a similar conclusion true if "open" is replaced by "closed"?

(4) Explain why the set of rational numbers x such that $|x| < \sqrt{2}$ is a closed subset of the rationals $\mathbb{Q}$ in the subspace topology.

(5) Let X be a space in which every one point subset is closed in X. Given a point $p \in X$, let $\mathcal{N}_p$ be the family of all open subsets in X containing p. If p and q are different points of X, why are $\mathcal{N}_p$ and $\mathcal{N}_q$ distinct families of subsets?

(6) Given a continuous function $f : X \to Y$, give an example such that X is Hausdorff but $f[X]$ is not.

(7) Let A be the subset of the coordinate plane consisting of all (x, y) such that either $x \geq 0$ or $y \geq 0$. Explain why A is connected.

(8) Describe all connected subsets A of the real line for which the interior of A is the open interval $(0, 1)$.

(9) Explain why the set of points (x, y) in the coordinate plane satisfying $|x| + |y| \leq 1$ is compact. [*Hint:* Why must we have $|x|, |y| \leq 1$, and why is the subset closed?]

(10) Explain why the set of points (x, y) in the coordinate plane satisfying $|x| \cdot |y| \leq 1$ is not compact.

(11) Let A be the set of all numbers $y = (3x + 4)/(5x + 6)$ where x ranges over all **positive** real numbers. What is the greatest lower bound of A? [*Hint:* Try graphing the function.]

(12) Let A be the set of all points (x, y) in the coordinate plane such that $y^2 = \pm 1$. Explain why A is not connected.

(13) Let A and B be bounded subsets of a metric space X. Explain why $A \cup B$ is also bounded, and if $A \cap B$ is nonempty give an upper estimate for its diameter involving the diameters of A and B. Also, give examples where this inequality fails if $A \cap B$ is empty.

(14) Let $f : X \to Y$ be a continuous function where X is compact and the topology on Y comes from some metric d. Explain why $f[X]$ is bounded, and if $y_0 \in Y$ explain why there is some point $x_0 \in X$ such that the distance function $d(f(x), y_0)$ takes a maximum value.

(15) Suppose that (X, d) is a connected metric space and u, v are distinct points of X. Prove that there is some point w such that $d(u, w) = \frac{1}{2}\, d(u, v)$.

(16) Suppose that (X, d) is a bounded connected metric space with diameter Δ. Prove that every number in $[0, \Delta)$ is the distance between two points of X.

(17) Let $A \subset \mathbb{R}^n$ be a bounded subset. Prove that every continuous real valued function $f : A \to \mathbb{R}$ is bounded and has a maximum value if and only if A is compact.

(18) Prove that there is a point $x \in \left(\frac{1}{2}\pi, \frac{3}{2}\pi\right)$ such that $\tan x = x$. [*Hint:* Graph the functions $f(x) = \tan x$ and $f(x) = x$. What are the limits of $\tan x - x$ as $x \to \frac{1}{2}\pi$ from the right and $x \to \frac{3}{2}\pi$ from the left?]

For Chapters 12 and 13, it is worthwhile to find examples where the criteria for recognizing connected and compact subspaces apply.

Course and examination grades

After the final examination has been graded, solutions and the grading curve will be posted for those who are interested in seeing them. My general policy is that students are welcome to take and keep their examinations. Students who want to retrieve their examinations should contact me by electronic mail next quarter so that arrangements can be made.