SOLUTIONS TO PROBLEMS FROM

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New problems

$\boxed{\text{Need } m \neq 0}$

1. (a) Let $h(x,y) = y - mx$, a continuous fcn.

$U_+ = h^{-1}[\text{positive reals}]$, $U_- = h^{-1}[\text{neg. reals}]$

both open in $\mathbb{R}$

so U_+ & U_- are open by continuity of h.

To see these sets are connected, first say $p, q \in U_+$. The set is convex $\Leftrightarrow$ for $0 \leq t \leq 1$ we have $(1-t)p + tq \in U_+$ all p, q.

Now $h(tu + (1-t)v) = t\,h(u) + (1-t)\,h(v)$ (verify this). But $p, q \in U_+$; $0 \leq t \leq 1 \Rightarrow$

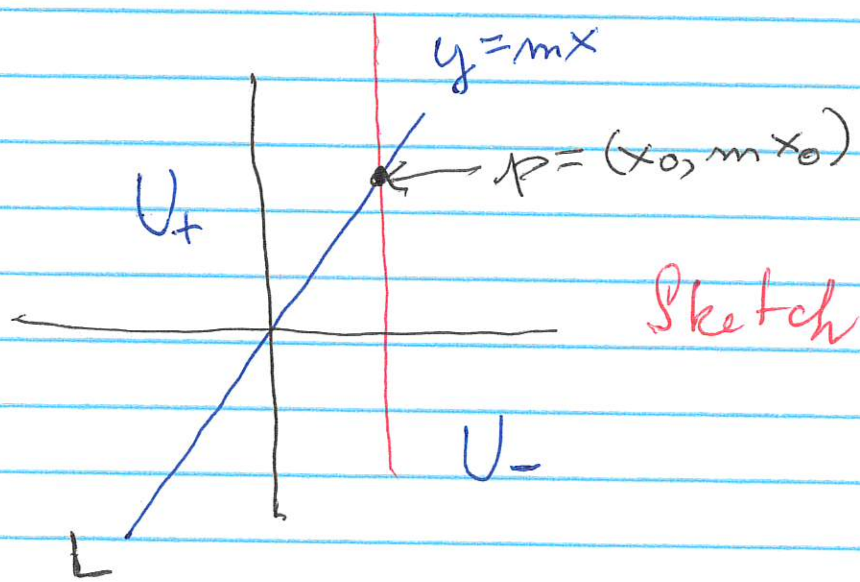
$$h((1-t)p + tq) = \underbrace{(1-t)}_{\text{nonneg}}\underbrace{h(p)}_{\text{pos}} + \underbrace{t}_{\text{nonneg}}\,\underbrace{h(q)}_{\text{pos}}$$

and at least one of t, $1-t$ is positive, so the right hand side is positive.

If $p, q \in U_-$ then $h(p), h(q) < 0$ and we need to show $h((1-t)p + tq) < 0$. Once again the left side is $(1-t)h(p) + t\,h(q)$. At least one of $t, 1-t$ is positive and both are nonnegative, so $(1-t)h(p) + t\,h(q) < 0$ as required.

(b) γ is supposed to be CONTINUOUS. Consider the continuous composite $h(\gamma(t))$. We have $h(\gamma(-1)) < 0 < h(\gamma(1))$ by assumption. The Intermediate Value Property for continuous real valued functions now implies that $h(\gamma(t_0)) = 0$ for some t_0. Now $p \in L \Leftrightarrow h(p) = 0$, so that $\gamma(t_0) \in L$.

(c)



Say $m > 0$. Then the sequence
$(x_0, mx_0 + \frac{1}{n})$ is in U_+ and
converges to (x_0, mx_0), and the sequence
$(x_0, mx_0 - \frac{1}{n})$ is in U_- and
converges to (x_0, mx_0). Hence (x_0, mx_0)
is a limit point of both U_+ and U_-.
If $m < 0$, then the sequences lie in U_-, U_+
instead of U_+, U_- respectively.

2. Say X and Y are T_1. If $p \in X$ + $q \in Y$
then $(p, q) \in X \times Y$. Now $\{p\}$ and $\{q\}$
are closed in X and Y respectively, and
hence $\{(p,q)\} = \{p\} \times \{q\}$ is closed
in $X \times Y$.

Now say X is T_1 and $a \in A \subseteq X$.
Then $\{a\}$ is closed in X and hence
$\{a\} = \{a\} \cap A$ is closed in A with respect
to the subspace topology.

3. FIRST Follow the comment. $\Delta_A \subseteq A \times A$ is closed since A is Hausdorff. Likewise $\Delta_B \subseteq B \times B$. Now $A \times A$, $B \times B$ are closed in $X \times X$ since A, B closed in X. Therefore Δ_A, Δ_B closed in $X \times X$, so that

$\Delta_X = \Delta_A \cup \Delta_B$ is also closed in $X \times X$.

Therefore X is Hausdorff.

4. SECOND <u>Claim</u> If $f : X \to Y$ is 1-1 continuous, and $A \subseteq X$, then $h[L_X(A)] \subseteq L_Y(h[A])$.

<u>PROOF.</u> Let $p \in L_X(A)$; want to show $h(x) \in L_Y(h[A])$. Let $h(x) \in V$ open, and choose U open so that $x \in U$ and $h[U] \subseteq V$. Since $x \in L(A)$, can find $y \in (U - \{x\}) \cap A$. But then $h(y) \in (h[U] - \{h(x)\}) \cap h[A] \subseteq (V - \{h(x)\}) \cap h[A]$, so that $h(y) \in L(h[A])$.

<u>APPLICATION TO PROBLEM.</u> We have

~~$h[L(A)] \subseteq L(h[A])$, so $L(A) = h^{-1}h[L(A)] \subseteq$~~

~~$h^{-1}[L(h[A])] \subseteq L(h^{-1}h[A]) = L(A)$, so that~~

$h[L(A)] \subseteq L(h[A])$ and also
$h^{-1}[L(h[A])] \subseteq L(h^{-1}h[A]) = L(A)$.
If we apply h we get $L(h[A]) \subseteq L(h[A])$,
so each set is a subset of the other.

4. If $C \subseteq X$ is connected, then so is $h[C] \subseteq Y$
so $h[C] \subseteq C'$ for some component of Y.
Likewise $C \subseteq h^{-1}[C']$ is an inclusion of connected sets. Since C is a maximal connected subset, $C = h^{-1}[C']$. Hence
$h[C] = hh^{-1}[C'] = C'$.

5. Proceed by induction on the number of subspaces.
One subspace — no problem.
Suppose true for a union of k subspaces and we are given $C_1, \ldots, C_{k+1}$ compact. By induction
$B = C_1 \cup \ldots \cup C_k$ is compact.

Let $\mathfrak{U}$ be an open covering of $C_1 \cup \ldots \cup C_k \cup C_{k+1}$
Then there are finite subcoverings $\mathfrak{V}$ of B
and $\mathfrak{W}$ of C_{k+1}. The union of $\mathfrak{V} + \mathfrak{W}$ defines a finite subcovering of $C_1 \cup \ldots \cup C_k \cup C_{k+1}$.

EXAMPLE. Let $X = \mathbb{Z}$ with the discrete topology. Then each set $\{n\}$ is closed and $X = \bigcup_n \{n\}$ is noncompact.

~~6. (a) Show f is continuous at every point p ∈ X.~~

6. (a) Let W be open in Y. Then

$(f|U)^{-1}[W] = U \cap f^{-1}[W]$ open in U, so open in X ①

$(f|V)^{-1}[W] = V \cap f^{-1}[W]$ open in V, so open in X. ②

$$\text{Therefore } f^{-1}[W] = \underbrace{(U \cap f^{-1}[W])}_{\text{open ①}} \cup \underbrace{(V \cap f^{-1}[W])}_{\text{open ②}}$$

$\Rightarrow f^{-1}[W]$ open in X.

(b) A similar result is true for closed subspaces. Replace "open" by "closed" throughout the preceding argument.

7. $p(1,0) = 1 > 0$ and $p(0,1) = -6 < 0$. Hence p takes every value between -6 and 1, for

① $[0,1] \times [0,1]$ is connected

② J = Image of $[0,1] \times [0,1]$ under p is too

③ J connected, $-6 + 1 \in J \Rightarrow [-6, 1] \subseteq J$.

8. It suffices to show $L(A)$ is compact. Now X metric $\Rightarrow L(A)$ closed. A compact $\subseteq X \Rightarrow A$ closed in X. Preceding two $\Rightarrow L(A)$ compact.

Older problems 1

Selected problems only. Some only sketched.

1. Case $n=2$ is in the notes. Let $B = A_1 \cup \dots \cup A_{n-1}$ & assume it is connected by induction. Then $B \cap A_n = \phi$. Apply the case in the first sentence.

2. (i) Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ be distinct points. Renumbering if necessary, we can assume $x_1 \in U$ open but $x_2 \notin U$, and $y_1 \in V$ open but $y_2 \notin V$. Then $(x_1, y_1) \in U \times V$ but $(x_2, y_2) \notin U \times V$. Now suppose $y_1 = y_2$. Then $(x_1, y) \in U \times Y$ but $(x_2, y) \notin U \times Y$. A similar argument works if $x_1 = x_2$ but $y_1 \neq y_2$.

4. (i) Limit points are all points with $x, y \geq 0$. If $x_0, y_0 \geq 0$, then $(x_0, y_0) = \lim_{n \to \infty} (x_0 + \frac{1}{n}, y_0 + \frac{1}{n})$.

(ii) Every ~~rational~~ open ray (c, ∞) contains a rational number, so the rational numbers are dense.

(iii) No set of the form $\mathbb{R} - (c, \infty) = (-\infty, c]$ is open in $\mathfrak{B}$.

5. (i) $A = E \cap U$ with E closed & U open
$B = F \cap V$ with F closed & V open $\Rightarrow$

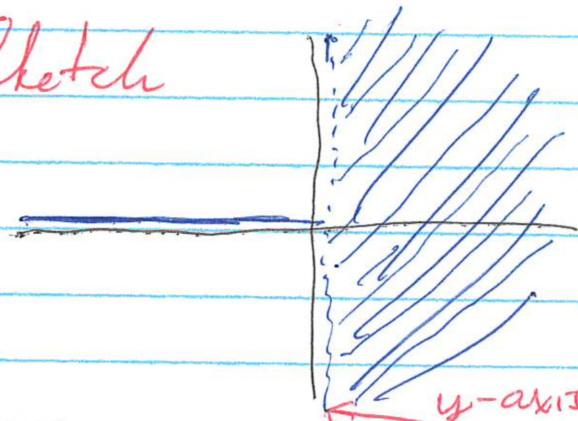
$A \cap B = (E \cap U) \cap (F \cap V) = \underbrace{(E \cap F)}_{\text{closed}} \cap \underbrace{(U \cap V)}_{\text{open}}$

$\Rightarrow A \cap B$ locally closed.

(ii) It is enough to show that all rectangular open sets $U \times V$ are unions of products $U_\alpha \times V_\beta$. This follows since $U = \bigcup_\alpha U_\alpha$, $V = \bigcup_\beta V_\beta \Rightarrow$ $U \times V = \bigcup_{\alpha, \beta} U_\alpha \times V_\beta$.

(iii) $\mathbb{R}-\{0\}$ is not open.

6. Sketch



$A = \mathbb{R}\times\{0\}$
$\cup$
$(0,\infty)\times\mathbb{R}$.

The interior is just $(0,\infty)\times\mathbb{R}$ and the boundary is $(\{0\}\times\mathbb{R}) \cup ([0,\infty)\times\{0\})$

Older problems 2

1. If $p\in X$ then for every open neighborhood U of p we know that $(U-\{p\})\cap X = U-\{p\}$ contains at least one point.

2. Let $(p,q)\in L(A)\times L(B)$, and suppose W is an open neighborhood of (p,q) in $X\times Y$. Take open neighborhoods U of p & V of q so that $U\times V\subseteq W$. Then there is $a\in(U-\{p\})\cap A$, $b\in(V-\{q\})\cap B$, and

therefore

$$(a,b) \in \left[(U-\{p\}) \cap A\right] \times \left[(V-\{q\}) \cap B\right] =$$

$$\left[(U-\{p\}) \times (V-\{q\})\right] \cap (A \times B) \subseteq$$

$$(W - \{(p,q)\}) \cap (A \times B).$$

4. The set is equal to $[-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q}$ since $\sqrt{2}$ is irrational.

5. $X - \{q\} \in \mathcal{N}_p$ and $X - \{p\} \in \mathcal{N}_q$.

6. f any function with # points in $|Y| \geq 2$ and f is onto, X discrete, Y indiscrete.

7. $A = ([0,\infty) \times \mathbb{R}) \cup (\mathbb{R} \times [0,\infty))$

all these spaces are connected.

Use the facts that products of connected spaces are connected and nondisjoint unions of connected (sub) spaces are connected.

8. $(0,1)$, $[0,1]$, $(0,1]$ and $[0,1)$. There can't be any points with $x<0$ or $x>1$ in such a set by the Intermediate Value Property.

9. Since $|x|+|y|$ is continuous, the set is closed. It is also bounded because

$|x|^2 = x^2$
$|y|^2 = y^2$

~~$|x|+|y| \leq 1 \Rightarrow$~~ $x^2+y^2 \leq |x|^2+2|x||y|+|y|^2 \leq 1$

This expression is $(|x|+|y|)^2$ Hence

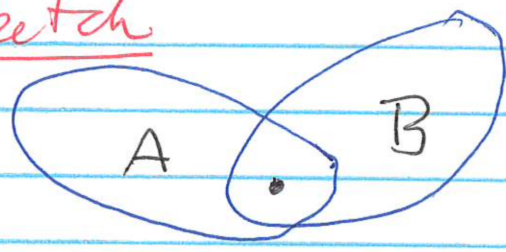
$|x|+|y| \leq 1 \Rightarrow \sqrt{x^2+y^2} \leq 1$, so that the set is bounded. Since the set is in $\mathbb{R}^2$, closed & bounded, it is compact.

10. This set is unbounded. It contains all points $(n, \frac{1}{n})$ where n runs through the positive integers.

11. We know that the limit as $x \to \infty$ is $\frac{3}{5}$, so it is enough to check the expression $> \frac{3}{5}$ for $x > 0$. Now $\frac{3x+4}{5x+6} > \frac{3}{5} \Leftrightarrow 15x+20 > 15x+18$ for $x>0$, and the second inequality is immediate (subtract $15x$ from both sides!).

12. A consists of the two horizontal lines $y=1$ and $y=-1$. Each line is closed in $\mathbb{R}^2 \Rightarrow$ closed in A, the lines are disjoint, and their union is A.

13. Sketch



$p \in A \cap B$

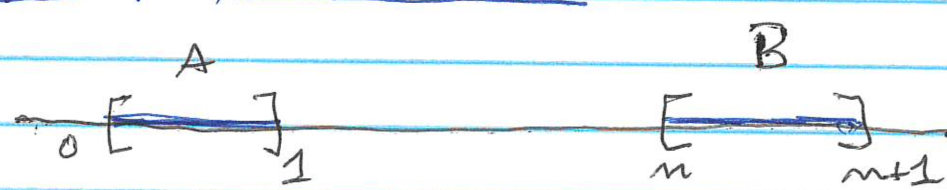
$d(x,y) \leq \text{diam}\, A + \text{diam}\, B$ if $x, y \in A$ or $x, y \in B$.

Now suppose (say) $x \in A$ & $y \in B$. Then

$$d(x,y) \leq d(x,p) + d(p,y) \leq \text{diam}\, A + \text{diam}\, B.$$

So $\text{diam}\,(A \cup B) \leq \text{diam}(A) + \text{diam}(B)$.

Example when $A \cap B = \phi$



$n > 1$

$\text{diam}\, A$, $\text{diam}\, B = 1$, but $\text{diam}\,(A \cup B) = n+1$.

14. $f[X]$ is compact ~~+~~ hence bounded.

Also $h(x) = d(f(x), y_0)$ is continuous $\Longrightarrow$ it takes a maximum value because X is compact.

15. Let $f(x) = d(u, x)$ so f is continuous. Since $f(u) = 0$ and X is connected, f takes every value in $[0, d(u,v)]$. In particular, $d(u,w) = f(w) = \frac{1}{2} d(u,v)$ for some w.

~~14. Much like 15, but here we want to show $d(p,q)$ for every $\varepsilon > 0$ there is some~~

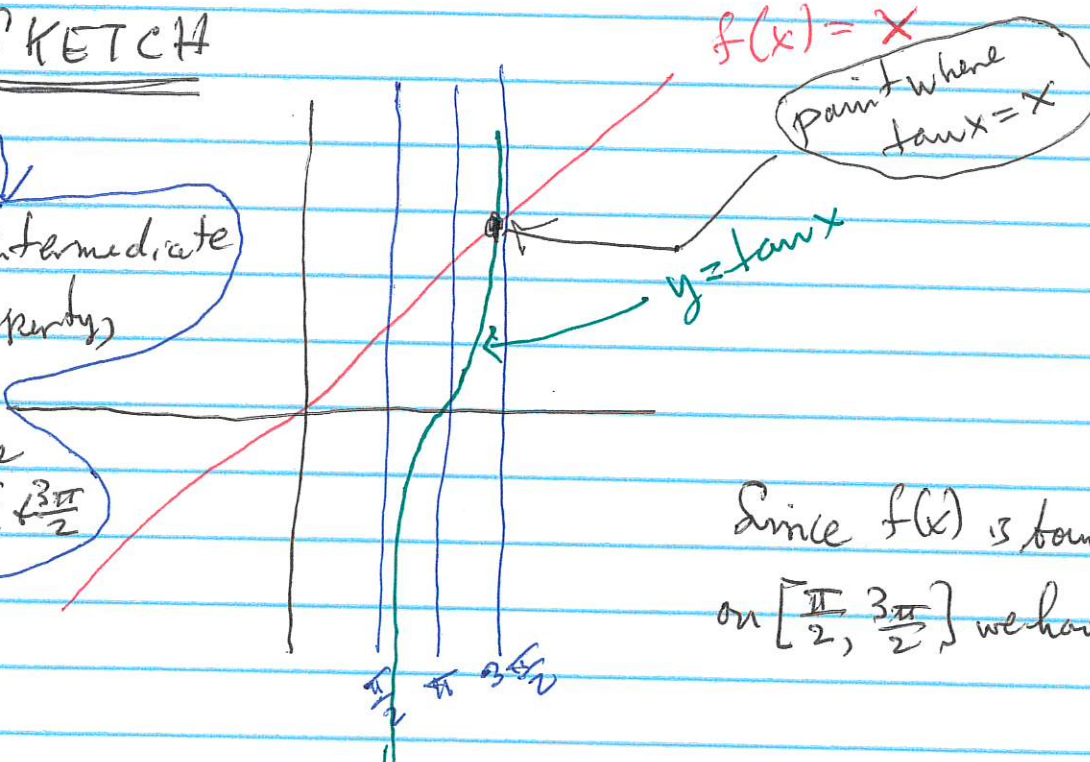
16. Let $\varepsilon > 0$ and $\varepsilon < \Delta$. Then there is a pair of points (p,q) so that $d(p,q) > \Delta - \varepsilon$ by definition of the diameter. Hence distance takes all values ~~between~~ in $[0, \Delta - \varepsilon]$ by the connectedness of X. Therefore the values for distance contain all ~~pos~~ numbers in $\bigcup_{0<\varepsilon<\Delta} [0, \Delta - \varepsilon] = [0, \Delta)$.

17. We know there is a maximum if A is compact. Suppose A is $\boxed{\text{not}}$ compact. Let $p \in L(A)$ but $p \notin A$; such a point exists because A is not closed. Then the function $f(a) = \dfrac{1}{|a-p|}$ is continuous on A.

Now if $a_n \in A$ and $\lim\limits_{n \to \infty} a_n = p$, then $\lim\limits_{n \to \infty} f(a_n) = \infty$, so f is not bounded on A.

18. SKETCH

By the intermediate value property, $\tan x = x$ somewhere between $\frac{\pi}{2}$ & $\frac{3\pi}{2}$



Since $f(x)$ is bounded on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ we have

$\lim\limits_{x \to \frac{\pi}{2}+} \tan x - x = -\infty$, $\lim\limits_{x \to \frac{3\pi}{2}-} \tan x - x = +\infty$

$\frac{d}{dx}$ ~~tan~~ $\tan x - x = \sec^2 x - 1 = \tan^2 x \geq 0$, positive if $x \neq \pi \Rightarrow$ fcn is strictly $\uparrow$ (increasing).