Compactness of closed intervals

Here is a detailed verification that a closed interval $[a, b]$ in the real line is compact; *i.e.*, every open covering $\mathcal{U} = \{U_\alpha\}$ has a finite subcovering.

Suppose that the interval $[a, b]$ and open covering $\mathcal{U}$ are given as in the preceding sentence. Let $x \in [a, b]$. Then there is some indexing value $\alpha(x)$ such that $x \in U_{\alpha(x)}$, and by the definition of the subspace topology there is some open interval J_x such that $J_x \cap [a, b] \subset U_{\alpha(x)}$.

By the Heine-Borel-Lebesgue Theorem, there is a finite subcollection of intervals $J_{x_1}, \cdots, J_{x_n}$ such that

$$[a, b] \quad \subset \quad J_{x_1} \cup \cdots \cup J_{x_n}$$

which means that

$$[a, b] \quad = \quad (J_{x_1} \cap [a, b]) \cup \cdots \cup (J_{x_n} \cap [a, b]) \ .$$

Since $J_{x_i} \cap [a, b] \subset U_{\alpha(x_i)}$ for all i and $U_\alpha \subset [a, b]$ for all α, this implies that

$$[a, b] \quad = \quad U_{\alpha(x_1)} \cup \cdots \cup U_{\alpha(x_n)}$$

and therefore a finite subcollection of $\mathcal{U}$ covers $[a, b]$. By the definition of compactness, it follows that $[a, b]$ is compact.∎