An example involving continuity

PROBLEM Let $a > 0$. Verify that $f(x) = \frac{1}{x}$ is continuous at $x = a$.

ANALYSIS We need to show that for each $\varepsilon > 0$ there is a $\delta > 0$ such that $|x - a| < \delta$ $\Longrightarrow \left|\frac{1}{x} - \frac{1}{a}\right| < \varepsilon$. Simplifying the left hand side, we need to show $\left|\frac{a-x}{xa}\right| < \varepsilon$ for $|x - a| <$ some δ, provided $x > 0$.

How does $\left|\frac{a-x}{xa}\right|$ depend on δ?

Let's suppose $\delta < a$ (otherwise we have problems with the condition $x > 0$.

We then have $\left|\frac{a-x}{xa}\right| < \frac{\delta}{|xa|} \leq \frac{\delta}{(a-\delta)\cdot a}$

So we need to answer the question, for which $\delta > 0$ is the right hand side $\leq \varepsilon$?

SOLUTION

First note we can restrict attention to finding δ such that $0 < \delta < B$ for some fixed bound B.

Now let $B = \frac{a}{2}$. Then $a - \delta > a - \frac{a}{2} = \frac{a}{2}$,

so $\dfrac{1}{a-\delta} < \dfrac{2}{a}$.

Therefore $\dfrac{\delta}{(a-\delta)a} \leq \dfrac{2\delta}{a^2}$.

Suppose now that we take $\delta < \frac{a}{2}$ and such that $\delta \leq \frac{a^2\varepsilon}{2}$. Then the preceding arithmetic shows that

$$\left|\frac{1}{x} - \frac{1}{a}\right| < \varepsilon \text{ if } |x-a| < \delta. \blacksquare$$