NAME: Answer Key

Mathematics 145A, Winter 2020, Examination 1+

Work all questions, and unless indicated otherwise give reasons for your answers. The point values for individual problems are indicated in brackets.

#	SCORE
1	
2	
3	
4	
TOTAL	

1. [25 *points*] Suppose that $A \subset \mathbb{R}$ is a nonempty bounded subset whose least upper bound is M, and let B be the set of all numbers of the form $a+1$ where a runs through all the elements of A. Prove that $M+1$ is the least upper bound of B. $N = \text{lub}(B)$.

First $M \leq N-1$, or $M+1 \leq N$.

If $a \in A$, then $a+1 \in B$ so $a+1 \leq N$ and $a \leq N-1$ all a. Hence $M = \text{l.u.b}(A) \leq N-1$.

Conversely, $b \in B \Rightarrow b-1 \in A$ and $b-1 \leq N-1$ all b. But $M = \text{lub}(A)$, so $b = a+1$ satisfies $b \leq M+1$. Since $N = \text{lub}(B)$ we have $N \leq M+1$.

Combining these, we see that $N = M+1$.

2. [25 *points*] Let (X, d_X) be a nonempty metric space, let d_p be any one of the product metrics on $X \times X$ (where $p = 1, 2, \infty$), and let $T(u, v) = (v, u)$ for $(u, v) \in X \times X$. Prove that T defines a continuous function from $(X \times X, d_p)$ to itself.

Let $\pi_1 : X \times X \longrightarrow X$ $\pi_1(x_1, x_2) = x_1$

$\pi_2 : X \times X \longrightarrow X$ $\pi_2(x_1, x_2) = x_2$.

It suffices to verify that $\pi_1 \circ T$ and $\pi_2 \circ T$ are continuous. But $\pi_1 \circ T = \pi_2$ & $\pi_2 \circ T = \pi_1$. Since π_1 & π_2 are continuous, so is T.

3. [25 *points*] Is the set $\mathbb{Q}$ of rational numbers an open subset of the real numbers $\mathbb{R}$? Either show why this is true or show why this is false.

False

Let $q \in \mathbb{Q}$ and $\varepsilon > 0$.

Claim $N_\varepsilon(q)$ contains irrational numbers.

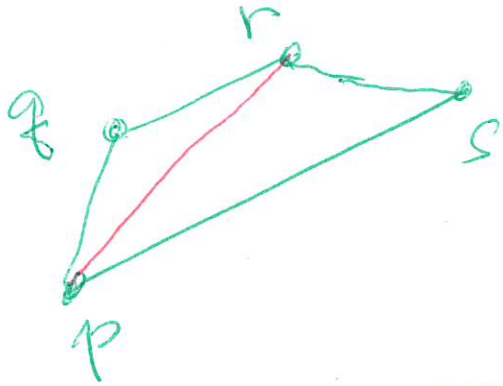
Proof Choose a positive integer n so $\frac{1}{n} < \varepsilon$. Then

$$q < q + \frac{\sqrt{2}}{2n} < q + \frac{1}{n}$$

So $N_{\frac{1}{n}}(q) \not\subseteq \mathbb{Q}$, and all the more so $N_\varepsilon(q) \not\subseteq \mathbb{Q}$.

4. [25 *points*] Let (X, d) be a metric space and let $p, q, r, s \in X$. Prove that

$$d(p,s) \;\leq\; d(p,q) \;+\; d(q,r) \;+\; d(r,s)\;.$$



Apply the Triangle Inequality twice.

$$d(p,r) \leq d(p,q) + d(q,r)$$
$$d(p,s) \leq d(p,r) + d(r,s)$$

Substituting the first inequality into the second, we obtain

$$d(p,r) \leq \big(d(p,q) + d(q,r)\big) + d(r,s)$$

which is what we wanted to verify.