Mathematics 145A, Winter 2014, Examination 1

Answer Key

1. [*20 points*] Suppose $f : X \to Y$ is a function of sets and $C \subset Y$.

(a) Prove that $f\big[\,f^{-1}[C]\,\big] \subset C$.

(b) If in the setting of (a) we have $f(x) = x^2$ and $C = [-1, 1]$, describe $C - f\big[\,f^{-1}[C]\,\big]$.

SOLUTION

(a) Suppose that y belongs to the set on the left hand side. Then $y = f(x)$ where $x \in f^{-1}[C]$, where the latter translates to $f(x) \in C$. Therefore we have $y = f(x)$ where $f(x) \in C$, and since $y = f(x)$ this means that $y \in C$.■

(b) In this case $f^{-1}[C] = [0, 1]$ and $f\big[\,f^{-1}[C]\,\big]$ is the image of $[0, 1]$ under $f(x) = x^2$. This image is $[0, 1]$ and hence the set $C - f\big[\,f^{-1}[C]\,\big]$ is equal to $[-1, 1] \; - \; [0, 1]$, which is the half-open interval $[-1, 0)$.■.

2. [*20 points*] Let f be a monotonically increasing real valued function on the open interval (a, b), let $c \in (a, b)$ and set $f(c-)$ equal to the least upper bound of all values $f(x)$ with $x < c$.

(a) Prove that $f(c-) \leq f(c)$.

(b) If $f(x) = 1$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$, evaluate $f(0) - f(0-)$.

SOLUTION

(a) Since f is monotonically increasing we have $f(x) \leq f(c)$ for all $x < c$; in other words $f(c)$ is an upper bound for the set of all values $f(x)$ with $x < c$. Since $f(c-)$ is the least upper bound of all such values, it follows that $f(c-) \leq f(c)$.■

(b) Since $f(0) = 0$ for $x < 0$, it follows that $f(0-) = 0$. By definition $f(0) = 1$, and therefore the difference $f(0) - f(0-)$ must be equal to 1.■

3. [*25 points*] Let (X, d) be a metric space, and let $x, y, z \in X$. Prove that $d(x, y) \geq |d(x, z) - d(y, z)|$. [*Hint:* Prove that the left hand side is greater than or equal to both $d(x, z) - d(y, z)$ and its negative.]

SOLUTION

The Triangle Inequality implies that

$$d(x, z) \ \leq \ d(x, y) + d(y, z) \quad \text{and} \quad d(y, z) \ \leq \ d(x, y) + d(x, z)$$

so that $d(x, y) - d(y, z) \leq d(x, z)$ and $d(y, z) - d(x, y) \leq d(x, z)$. Since $|d(x, z) - d(y, z)|$ is the larger of the two expressions on the left hand sides of these inequalities, it follows that $d(x, y) \geq |d(x, z) - d(y, z)|$.■

4. [*15 points*] Let (X, d) be a metric space, and let $p \in X$. Prove that $X - \{p\}$ is an open subset of X. [*Hint:* If $y \neq p$, consider $N_r(y)$ where $r = d(p, y)$. If $d(z, y) < r$, is $z = p$ possible?]

SOLUTION

Follow the hint. If $d(z, y) < r = d(p, y)$ then p and y cannot be equal. But this means that $z \in X - \{p\}$. In other words, $N_r(y)$ is contained in $X - \{p\}$, which is the defining condition for a subset to be open in a metric space.■

5. [*20 points*] Describe explicit subsets A, B of the real line which satisfy the following conditions:

(a) Neither is open, but $A \cap B$ is open.

(b) Neither is open, but $A \cup B$ is open.

(d) Both A and $A \cup B$ are open, but B is not open.

(d) Both A and $A \cap B$ are open, but B is not open.

[*Hint:* Take A and B to be intervals which might be open, closed or half-open.]

SOLUTION

These examples are definitely not unique.

(a) Take $A = [-1, 1)$ and $B = (-1, 1]$, so that $A \cap B = (-1, 1)$.■

(b) Take $A = [0, 1)$ and $B = (-1, 0]$, so that $A \cup B = (-1, 1)$.■

(c) Take $A = (0, 2)$ and $B = (-1, 1]$, so that $A \cup B = (-1, 2)$.■

(d) Take $A = (-1, 1)$ and $B = (0, 2]$, so that $A \cap B = (0, 1)$.■