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# Mathematics 145A, Winter 2016, Examination 1

## Answer Key

1. [*25 points*] If $(X, d)$ is a metric space, then $A \subset X$ is said to be **locally closed** in $X$ if $A = E \cap V$, where $E$ is closed in $X$ and $V$ is open in $X$.

$(i)$ Suppose that $f : (X_1, d_1) \to (X_2, d_2)$ is a continuous mapping and $A \subset X_2$ is locally closed. Prove that $f^{-1}[A]$ is a locally closed subset of $X_1$.

$(ii)$ Suppose that $A$ and $B$ are locally closed in $X$, where as usual $(X, d)$ is a metric space. Prove that $A \cap B$ is also locally closed in $X$.

SOLUTION

$(i)$ Write $A = E \cap V$ where $E$ is closed in $X_2$ and $V$ is open in $X_2$. Then we have

$$f^{-1}[A] \quad = \quad f^{-1}[E \cap V] \quad = \quad f^{-1}[E] \cap f^{-1}[V] \ .$$

By continuity $f^{-1}[E]$ is closed in $X_1$ and $f^{-1}[V]$ is open in $X_1$, and therefore the inverse image of $A$ is locally closed in $X_1$.■

$(ii)$ Write $A = E \cap V$ where $E$ is closed in $X_2$ and $V$ is open in $X_2$, and also write $B = F \cap W$ where $F$ is closed in $X_2$ and $W$ is open in $X_2$. Then we have

$$A \cap B \quad = \quad (E \cap V) \ \cap \ (F \cap W) \quad = \quad (E \cap F) \ \cap \ (V \cap W) \ .$$

Since $E \cap F$ is closed in $X$ and $V \cap W$ is open in $X$, it follows that $A \cap B$ is locally closed in $X$.■

2. [*25 points*] Suppose that $(X, d)$ is a metric space and that $x, y, z \in X$ satisfy $d(x, z) \leq \frac{1}{2}d(x, y)$. Prove that $d(y, z) \geq \frac{1}{2}d(x, y)$.

SOLUTION

By the Triangle Inequality we have $d(x, y) \leq d(x, z) + d(y, z)$, which implies $d(y, z) \geq d(x, y) - d(x, z)$. Since $d(x, z) \leq \frac{1}{2}d(x, y)$, it follows that the right hand side is greater than or equal to $x(x, y) - \frac{1}{2}d(x, y) = \frac{1}{2}d(x, y)$.$\blacksquare$

3. [*25 points*] Let $(Y, \Delta)$ be a metric space with the usual discrete metric, let $(X, d)$ be an arbitrary metric space, and let $f : Y \to X$ be a map of sets. Prove that $f$ defines a continuous mapping from $(Y, \Delta)$ to $(X, d)$.

SOLUTION

We need to show that if $V$ is an open subset in $(X, d)$, then its inverse image $f^{-1}[V]$ in $(Y, \Delta)$ is also open. However, **every** subset in $Y$ is open with respect to the discrete metric, and therefore $f^{-1}[V]$ is automatically open, which means that $f$ is automatically continuous.■

4. [*25 points*] Let $(X, d)$ be a metric space, and let $d'(x_1, x_2) = 100\, d(x_1, x_2)$. Prove that $d'$ is also a metric on $X$.

SOLUTION

We have $d'(x,y) = 100d(x,y)$ and this is nonnegative because $d$ is nonnegative. If $0 = d'(x,y) = 100d(x,y)$, then it follows that $d(x,y) - 0$ and hence $x = y$. Also $d'(y,x) = 100d(y,x)$, and since $d$ is a metric the right hand side is equal to $100d(x,y) = d'(x,y)$; therefore d' is symmetric in $x$ and $y$. Finally, we may check the Triangle Inequality as follows:

$$d'(x,y) \quad = \quad 100d(x,y) \quad \leq \quad 100\big(d(x,z) + d(y,z)\big) \quad =$$

$$100d(x,z) \;+\; 100d(y,z) \quad = \quad d'(x,z) \;+\; d'(y,z)\ .$$

Therefore $d$ satisfies all the properties required of a metric.∎