Mathematics 145A, Winter 2017, Examination 1

Answer Key

1. [20 *points*] Suppose that (X, d) is a metric space, let $a \in X$, let $\delta > 0$, and let $N_\delta(a)$ be the open disk of radius δ centered at a; in other words, $N_\delta(a)$ is the set of all $x \in X$ such that $d(a, x) < \delta$. Prove that $N_\delta(a)$ is an open subset of X.

SOLUTION

Suppose that $x \in N_\delta(a)$ and let $r = \delta - d(a, x)$; the right hand side is positive because $x \in N_\delta(a)$ implies $d(a, x) < \delta$. Consider the open disk $N_r(x)$; we claim that $N_r(x) \subset N_\delta(a)$. This follows because $y \in N_r(x)$ implies $d(x, y) < r$, so that

$$d(y, a) \leq d(y, x) + d(x, a) < r + d(x, a) =$$

$$(\delta - d(a, x)) + d(a, x) = \delta$$

so that $N_r(x) \subset N_\delta(a)$.■

2. [30 *points*] (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let A be the half-open interval $(-2, 2]$. Prove that $f^{-1}[A]$ is the intersection of an open subset of $\mathbb{R}$ with a closed subset of $\mathbb{R}$. [*Hint:* Explain why A also has this property.]

(b) In the preceding, specialize to the function $f(x) = \sin x$. In this case determine whether $f^{-1}[A]$ open, closed, both or neither, and give reasons for your answer. [*Hint:* What is $f[\mathbb{R}]$?]

SOLUTION

(a) The half-open interval is the intersection of the open set $V = (-2, \infty)$ and the closed set $E = (-\infty, 2]$. Therefore we have

$$f^{-1}[A] \quad = \quad f^{-1}[V \cap E] \;\; f^{-1}[V] \cap f^{-1}[E]$$

and by the continuity of f we know that $f^{-1}[V]$ is open and $f^{-1}[E]$ is closed (in $\mathbb{R}$).■

(b) The image of the sine function is the closed interval $\Delta = [-1, 1]$, which is contained in A. Therefore $f^{-1}[A] \supset f^{-1}[\Delta] = \mathbb{R}$, and since the left hand side is a subset of $\mathbb{R}$ by definition, we have $f^{-1}[A] = \mathbb{R}$. Since $\mathbb{R}$ is both an open and a closed subset of itself, it follows that $f^{-1}[A]$ is both open and closed.■

3. [*25 points*] Let X be a set, and let d and d' be metrics on X. Prove that the sum $d+d'$ is also a metric. [*Hint:* If $u, v \geq 0$ and $u+v=0$, what can we say about u and v?]

SOLUTION

If $u \geq 0$ and $v \geq 0$, then $u+v=0$ forces u and v to be both zero; if either were positive then the sum would be positive. Knowing this we may verify the conditions for a metric as follows:

> Since $d \geq 0$ and $d' \geq 0$ we must have $d+d' = 0$. Furthermore, if $d(x,y)+d'(x,y) = 0$, then the preceding paragraph implies that $d(x,y) = 0 = d'(x,y)$, so that $x = y$.
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> By the symmetry property of metrics we have $d(y,x)+d'(y,x) = d(x,y)+d'(x,y)$.
>
> By the Triangle Inequality for d and d' we have $d(x,y) + d'(x,y) \leq [d(x,z) + d(z,y)] + [d'(x,z) + d'(z,y)] = [d(x,z) + d'(x,z)] + [d(z,y) + d'(z,y)]$.■

4. [25 *points*] (a) Given a continuous function $f : X \to Y$ (where X and Y are metric spaces) the graph of $\Gamma_f : X \to X \times Y$ is defined by $\Gamma_f(x) = (x, f(x))$. Prove that Γ_f is continuous.

(b) If X and Y are metric spaces, a function $f : X \to Y$ is said to be *nonexpanding* if for all $x_1, x_2 \in X$ we have $d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2)$. Prove that such a function is continuous.

SOLUTION

(a) It suffices to show that the coordinate functions $\pi_X \circ \Gamma_f$ and $\pi_Y \circ \Gamma_f$ are continuous, where π_X and π_Y are the coordinate projections onto X and Y respectively. These follow immediately because $\pi_X \circ \Gamma_f = \mathrm{id}_X$ and $\pi_Y \circ \Gamma_f = f$.■

(b) In fact the function is uniformly continuous, for if $\varepsilon > 0$ and $\delta = \varepsilon$, then $d(u, v) < \delta$ implies $d(f(u), f(v)) \leq d(u, v) < \delta = \varepsilon$; in other words, we can take $\delta(\varepsilon)$ to be ε itself.■