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Mathematics 145A, Winter 2019, Examination 1

Answer Key

1. [25 *points*] Let (X, d) be a metric space, and let $p \in X$. Show that $X - \{p\}$ is an open subset of X.

SOLUTION

Suppose that $q \neq p$ and $q \in X$, which is equivalent to the statement $q \in X - \{p\}$. Let $a = d(p, q)$ and set $r = \frac{1}{2}a$. We claim that $N_r(q) \subset X - \{p\}$; in other words, if $q \neq p$ and $x \in N_r(q)$ then $x \neq p$. This follows immediately since $d(x, q) < \frac{1}{2}a$ and $d(p, q) = a$.■

2. [20 *points*] Let (X, d) and(Y, d') be metric spaces. A function $F : X \to Y$ is said to satisfy a Lipschitz condition if there is a constant $K > 0$ such that $d'(F(u), F(v)) \leq K \cdot d(u, v)$ for all $u, v \in X$. Show that F is continuous if it satisfies such a condition.

SOLUTION

Let $\varepsilon > 0$, let $x \in X$ and take $\delta = \varepsilon/K$. Then $d(v, x) < \delta$ implies

$$d'(F(x), F(v)) \quad \leq \quad K\delta \quad = \quad K \cdot \frac{\varepsilon}{K} \quad = \quad \varepsilon$$

and hence F is continuous at x. Since x was arbitrary, F is continuous everywhere in X.■

3. [*25 points*] Let (X, d) be a metric space, and let A be a subset of X which is neither closed nor open. Prove that the relative complement $X - A$ is also neither closed nor open.

SOLUTION

Since a set is open if and only if its complement is closed, we and A is not closed, it follows that $X - A$ is not open. Similarly, since A is not open, it follows that $X - A$ is not closed.■

4. [30 *points*] (a) Give an example of a closed subset $A \subset \mathbb{R}$ and a point $a \in A$ such that a is not a limit point of A.

(b) Let Y_+ and Y_- be the sets of positive and negative real numbers respectively. Show that the boundary of each set is equal to $\{0\}$.

SOLUTION

(a) Take A to be any one point subset $\{a\}$. Then $a \notin L(A)$ because for all $\varepsilon > 0$ we know that $(N_\varepsilon(a) - \{a\}) \cap A$ is a subset of $A - \{a\}$, which is empty.■

(b) Let's take the description of the boundary of a set E as $\overline{E} - \mathrm{Int}\,(E)$. Both Y_+ and Y_- are open subsets of $\mathbb{R}$ so the boundaries are just $\overline{Y_+} - Y_+$ and $\overline{Y_-} - Y_-$. Neither Y_+ nor Y_- is closed in $\mathbb{R}$, but these subsets are contained in the closed subsets $[0, +\infty)$ and $(-\infty, 0]$ respectively, and therefore the latter are the closures of the respective subsets. Consequently, for both these subsets E the complement of E in $\overline{E}$ is just $\{0\}$.■