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Mathematics 145A, Winter 2020, Examination 1

Answer Key

1. [25 *points*] Suppose that $A \subset \mathbb{R}$ is a nonempty bounded subset whose least upper bound is M, and let $2A$ be the set of all numbers of the form $2a$ where a runs through all the elements of A. Prove that $2M$ is the least upper bound of $2A$.

SOLUTION

Since $a \in A$ implies $a \leq M$ we know that $2a \leq 2M$ for all a and hence $2M$ is an upper bound for A. Suppose now that N is the least upper bound for $2A$; by the preceding, we have $N \leq 2M$. Then $2a \leq N$ for all a and hence $a \leq \frac{1}{2}N$ for all $a \in A$. Therefore $\frac{1}{2}N$ is an upper bound for A and hence $M \leq \frac{1}{2}N$. If we combine the two inequalities for M and N we see that $N = 2M$.■

2. [25 *points*] Let $f : \mathbb{R} \to \mathbb{R}$ be a function which need not be continuous, let d be the standard metric on $\mathbb{R}$, and define a new metric d^* on $\mathbb{R}$ by $d^*(u,v) = |u-v|+|f(u)-f(v)|$. Verify that the "identity" mapping $j : (\mathbb{R}, d^*) \to (\mathbb{R}, d)$, defined by $j(x) = x$, is a continuous mapping of metric spaces. [*Hint:* Given $\varepsilon > 0$, one can find a corresponding $\delta > 0$ which is the same for all choices of x.]

SOLUTION

Following the hint, we want a $\delta > 0$ such that for all x we have $|u - v| < \varepsilon$ whenever $|u - v| + |f(u) - f(v)| < \delta$. Since the summands on the right hand side are nonnegative we have

$$|u - v| \quad \leq \quad |u - v| + |f(u) - f(v)| \quad = \quad d^*(u, v)$$

so $|u - v| < \varepsilon$ if the right hand side is less than ε. But this means we can take $\delta = \varepsilon$.

3. [25 *points*] Given the two subsets $\{0\}$ and $\mathbb{R} - \{0\}$ of the real numbers, one is open and one is not. Determine which is open and which is not open, and for each subset give reasons for your answer.

SOLUTION

If $\{0\}$ were open in the real line then some interval of the form $(-h, h)$ would be contained in $\{0\}$. Sine $h/2 > 0$ belongs to this interval, there is no interval of the given form with the desired property, and therefore $\{0\}$ cannnot be an open subset of the real line.

To show the complement is open, for each $x \neq 0$ we need to construct an open interval of the form $(x - h, x + h)$ which is entirely contained in $\mathbb{R} - \{0\}$. In fact there are unique intervals where h is as large as possible: If $x > 0$ this is the interval $(0, 2x)$, while if $x < 0$ this is the interval $(2x, 0)$.■

4. [25 *points*] Suppose that (X, d) is a metric space and that $x, y, z \in X$ satisfy $d(x,z) \leq \frac{1}{4} d(x,y)$. Prove that $d(y,z) \geq \frac{3}{4} d(x,y)$.

SOLUTION

By the Triangle Inequality and symmetry property for metrics we have

$$d(x,y) \quad \leq \quad d(x,z) \ + \ d(y,z)$$

and if we substitute the assumed inequality we also have

$$d(x,y) \quad \leq \quad \frac{1}{4} d(x,y) \ + \ d(y,z) \ .$$

If we subtract the nonnegative quantity $\frac{1}{4} d(x,y)$ from both sides of this equality we find that $d(y,z) \geq \frac{3}{4} d(x,y)$.■