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Mathematics 145A, Winter 2020, Examination 2

Answer Key

1. [*25 points*] Let X be a topological space, and let A, B, C, D be connected subsets of X such that $A \cap B$, $B \cap C$ and $C \cap D$ are all nonempty. Prove that $A \cup B \cup C \cup D$ is connected.

SOLUTION

Since $A \cap B \neq \emptyset$, the union $A \cup B$ is connected. Likewise, since $\emptyset \neq B \cap C \subset (A \cup B) \cap C$, we know that $A \cup B \cup C$ is connected. Finally, since $\emptyset \neq C \cap D \subset (A \cup B \cup C) \cap D$, it follows that $A \cup B \cup C \cup D$ is connected.■

2. [*25 points*] Let (X, d) be a metric space. Given $\varepsilon > 0$ and $x \in X$ define the closed neighborhood $CN_\varepsilon(x)$ to be the set of all $y \in X$ such that $d(x, y) \leq \varepsilon$. General considerations imply that this set contains the closure $\overline{N_\varepsilon(x)}$ of the open neighborhood $N_\varepsilon(x)$. Give examples of metric spaces where (a) these two sets are equal, (b) these two sets are unequal. [*Hint:* For the second one, there are simple standard subsets Y of the real line where $N_\varepsilon(x; Y)$ is a closed set.]

SOLUTION

(a) If X is the closed interval $[\varepsilon, \varepsilon]$ and $x = 0$, then $N_\varepsilon(x) = (-\varepsilon, \varepsilon)$ and its closure is $[-\varepsilon, \varepsilon] = CN_\varepsilon(x) = X$. (b) If X is the subset of the real line given by $\{-\varepsilon, 0, \varepsilon\}$ and $x = 0$, then $N_\varepsilon(x)$ is just the closed set $\{0\}$ but $CN_\varepsilon(x)$ is once again all of X.■

3. [*25 points*] If $\mathbb{Z}$ and $\mathbb{Q}$ are the integers and rational numbers respectively and $\mathbf{L}(\mathbb{Z})$, $\mathbf{L}(\mathbb{Q})$ denote their sets of limit points in the real numbers $\mathbb{R}$, then one of these sets is empty and the other is all of $\mathbb{R}$. State which one is empty and which is $\mathbb{R}$, and verify your assertion for either $\mathbf{L}(\mathbb{Z})$ or $\mathbf{L}(\mathbb{Q})$; you need not verify the other one.

SOLUTION

The set $\mathbb{Z}$ has no limit points, and the limit point set for $\mathbb{Q}$ is all of $\mathbb{R}$.

To verify the first assertion, notice that if n is an integer and U is the open neighborhood $(n-1, n+1)$ then $(U - \{n\}) \cap \mathbb{Z} = \emptyset$, so the criterion for n to be a limit point of $\mathbb{Z}$ fails to be true.■

To verify the second assertion, let $x \in \mathbb{R}$, let $x \in U$ open, and choose $h > 0$ so that $(x - h, x + h) \in U$. Then we know that there are rational numbers $p \in (x - h, 0)$ and $q \in (0, x + h)$, so by construction p and q lie in the intersection $(U - \{x\}) \cap \mathbb{Q}$ and hence the latter is nonempty. The latter shows that x is a limit point of $\mathbb{Q}$.■

According to the problem, only one of these verifications is required.

4. [*25 points*] Suppose that X is a topological space with at least two points, and for every pair of distinct points $p \neq q$ in X there is a continuous a continuous function $f : X \to [0,1]$ such that $f(p) = 0$ and $f(q) = 1$. Prove that X satisfies the Hausdorff Separation Property.

SOLUTION

If U and V are the inverse images of $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$ respectively, then these sets are open and disjoint subsets of X Furthermore, $p \in U$ because $f(p) = 0$ and $q \in V$ because $f(q) = 1$, so that U and V are disjoint open neighborhoods containing p and q respectively. Therefore X is a Hausdorff space by the definition of the Hausdorff Separation Property.■

5. [*25 points*] Let X be a compact topological space. Prove X satisfies the following *Ascending Chain Condition* for open subsets:

> If we are given a sequence of open subsets $U_1 \subset U_2 \subset U_3 \subset \ \dots$ such that $X = \cup_n \, U_n$, then there is some M such that $k \geq M$ implies $U_k = U_M = X$.

[*Hint:* Recall the proof that a compact metric space is bounded.]

SOLUTION

By construction the sets U_n form an open covering of X and hence there is a finite subcovering $U_{i(1)}, \ \cdots \ , U_{i(k)}$. Let M be the largest of the indices $i(j)$. Then by the assumption $U_1 \subset U_2 \subset U_3 \subset \ \dots$ we know that U_M contains all the other sets, so that U_M must be equal to X. Finally, since we have an increasing sequence of open sets in X it follows that $X = U_M = U_k$ for $k \geq M$.■

6. [*25 points*] Suppose that (X, d) is a connected metric space with at least two points. Prove that there is a continuous real valued function f whose image contains a closed interval $[0, r]$ for some $r > 0$. [*Hint:* The metric d is a continuous function from $X \times X$ to $\mathbb{R}$.]

SOLUTION

Let $x_0 \in X$ be fixed. As indicated by the hint, $f(y) = d(y, x_0)$ defines a continuous real valued function on X. Since X is connected we know that $f[X] \subset \mathbb{R}$ is also connected and hence is an interval. This interval contains $0 = f(x_0)$ and $r = f(y_0) > 0$ where y_0 is some point of $X - \{x_0\}$. Since $f[X]$ is connected it follows that this set must also contain the closed interval $[0, r]$.■